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"MEET DOCTOR FRANKLIN." *

BY

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Mr. Staples, Ladies and Gentlemen: Since I have at one time or other spoken about so many aspects of Franklin to so many groups in Philadelphia I may not be able to avoid some repetitions this afternoon, but I will do what I can to avoid it. At this opening talk of this interesting series I shall have to be a little more discursive than some of the more systematic speakers who come later.

Franklin had, I think, the most eminent mind that has ever existed in America. No wonder there are so many legendary misconceptions of him that it is difficult now to restore and comprehend him in the great integrity of his mind, character, and personality. He appears, somehow, to be a syndicate of men. We study him as a scientist, as a diplomat, as a statesman, as a business man, as an economist, as a printer, as a humorist and wit, as a great writer, as a sage, and as a landmark in the history of human speech about the common ways of life. What had been said before he so often said better. He was great in friendship, and in his later years was probably the most renowned private citizen on

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earth. It has recently been more than once remarked—and printed—that Benjamin Franklin was the American Leonardo di Vinci. This is American modesty, if not colonialism. Why not occasionally say that Leonardo di Vinci was the Italian Benjamin Franklin?

Franklin was the earliest American whom, without limiting ourselves to national terms, we can call a very great man. When you try to define his particular greatness you run into what John Adams felt in his jealous days in Paris. Adams was a great man, but not a very great man. A great man such as Adams living with a very great man such as Franklin cannot tell the difference between himself and the other. Adams could not tell why people thought the difference to be so enormous. Nor can anybody express the difference better than with a fairly common, if not entirely just, geographical analogy. Imagine yourself in a range of mountains. You look up, and several of the mountains seem to you, from where you stand, as high as the master of them all. But when you get to a distance which gives perspective you see that the great mountain towers above the others, which are, in a sense, only foothills to it. This was true of many men with whom Franklin was associated during his life, who were as far from being aware of his genuine superiority as posterity has sometimes been since then. Moreover, his superiority is the hardest kind to measure. We can sometimes measure superiority when it shows itself in outward acts, achievements, tumults, benefits, or damages. Franklin's eminence was in his almost supreme mind that moved to its countless tasks with what seems perfect ease. Both the supremacy and the ease are hard for us to explain because they are so nearly unique in history that comparisons fail us. And without comparisons there can be no measurements.

There are, of course, people who take an attitude toward Franklin that may remind us of the fate of Aristides. For his virtues, you will remember, he had a peevish vote cast against him by an illiterate man who was tired of hearing Aristides called Aristides the Just. We often have an impulse to fear and distrust great excellence. We say we like it, but in our hearts we suspect it of being more—and consequently less—than human. When we say this or that hero

is "human" we always mean he is weak in some way that is comforting. All of us are weak, and we like to believe this is human of us. When we find a similar weakness in a great man we are pleased because it means that the great are not so much greater than we are after all.

It is now and then asked if Franklin did not get most or many of his ideas from other men, and then out of vanity take the credit to himself. I do not think Franklin was particularly vain. In the third paragraph of his Autobiography he disarmingly admitted that telling his story might gratify his vanity. "Most people dislike vanity in others, whatever share they have of it themselves; but I give it quarter wherever I meet with it, being persuaded that it is often productive of good to its possessor, and to others that are within his sphere of action; and therefore, in many cases, it would not be altogether absurd if a man were to thank God for his vanity among the other comforts of life." Franklin knew that the false modesty which men conventionally affect is a mode of self-conscious egotism.

What could Franklin do when he wrote his Autobiography but tell of things he had done or helped do? Why should he have talked as with his hand over his mouth or his elbow over his head? Everybody knew he was great and famous. For him to pretend not to know that would have been silly, a form of stage fright. In a letter about the classic epigram of Turgot which said Franklin had snatched the lightning from the sky and the scepter from tyrants, Franklin in 1781 honestly protested. "It ascribes too much to me, especially in what relates to the tyrant; the Revolution having been the work of many able and brave men, wherein it is sufficient honor for me if I am allowed a small share."

Or go back to an earlier year in Franklin's life, when he was not so famous and might have been tempted not to be so generous. In the fall of 1753 he got together what he called a "philosophical packet" of letters exchanged between him and various scientific friends of his in America. It was an important year for Franklin. The king of France had complimented the remote Philadelphia tradesman on his electrical discoveries, Harvard and Yale had given him honorary degrees, the Royal Society in November awarded him the

Copley gold medal. The "philosophical packet" was to signalize what Franklin hoped would be a fresh beginning for him in the scientific career he looked forward to. Among these papers was a letter from James Bowdoin of Boston, dated November 12, in which he—before any other scientist so far as is known—hit on the first true explanation of "luminosity" (phosphorescence) in sea water: "that the said appearance might be caused by a great number of little animals, floating on the surface of the sea, which, on being disturbed, might, by expanding their finns, or otherwise moving themselves, expose such a part of their bodies as exhibits a luminous appearance, somewhat in the manner of a glow-worm, or fire-fly."

Here was a subject Franklin had already speculated on. He had at first conjectured that this luminosity was "owing to electric fire, produced by friction between the particles of water and those of salt. Living far from the sea, I had then no opportunity of making experiments on the sea water, and so embraced this opinion too hastily. For in 1750 and 1751, being occasionally on the sea coast, I found, by experiments, that sea water in a bottle, though at first it would by agitation appear luminous, yet in a few hours it lost that virtue; hence, and from this . . . I first began to doubt of my former hypothesis, and to suspect that the luminous appearance in sea water must be owing to some other principles."

Bowdoin's conjecture at once struck Franklin as sounder than his own. "It is indeed very possible," he wrote in a letter dated 13 December, "that an extremely small animalcule, too small to be visible even by the best glasses, may yet give a visible light. I remember to have taken notice, in a drop of kennel water magnified by the solar microscope to the bigness of a cart-wheel, there were numbers of visible animalcules swimming about; but I was sure there were likewise some which I could not see, even with that magnifier; for the wake they made in swimming to and fro was very visible, though the body that made it was not so. Now, if I could see the wake of an invisible animalcule, I imagine I might much more easily see its light if it were of the luminous kind. For how small is the extent of a ship's wake, compared with that of the light of her lantern."

Dr. Edwin Grant Conklin of the American Philosophical Society recently told me that a Japanese scholar had called Franklin's comment on these animalcules in sea water the earliest guess at the existence and nature of the microörganisms, as we should now call them, which are responsible for the phenomenon. Franklin had nothing to do with the error. He kept back his own letter (to be published long after his death), and sent Bowdoin's to London, where it was read before the Royal Society in December 1756 and later included in the 1769 edition of Franklin's "Experiments and Observations on Electricity," as "a Letter from J. B. Esq; in Boston, to B. F. concerning the Light in Sea-Water." The credit was Bowdoin's, and Franklin plainly gave it to him.

This same 1769 volume, among the most fascinating in the whole range of eighteenth-century "philosophical" writings, punctiliously credited Thomas Hopkinson, Ebenezer Kinnersley, and Philip Syng with their original suggestions and discoveries made during the "Philadelphia experiments." If the three came to be overlooked in the light of their greater colleague, it was hardly Franklin's fault. Reporting to Peter Collinson in London, Franklin constantly spoke of "our electrical enquiries" and the things "we" had found out, never pretending to have done the work alone. The first collection of his "Experiments and Observations" (1751) was possibly printed in London before Franklin in Philadelphia had known it was to be. Because he alone had written the reports, his name alone was given on the title-page. In his own copy of the pamphlet he marked each experiment with the initials of the discoverer: Hopkinson's once, Kinnersley's seven times and once jointly with Franklin's, Syng's three times. There can be little doubt that Franklin was always chief among the experimenters, as there can be none that he was the best writer among them. But he did not make excessive claims for himself. What happened was that as he went on growing more and more famous, the world credited him with more and more achievements, past as well as present. In the world at large, first men are dramatically first, and the rest nowhere to speak of.

I know it is commonly said that the Autobiography is a retouched photograph in which Franklin emphasizes his own

share in the life of his times and minimizes that of other men. I also know that I have spent many days and weeks investigating this very matter and have come to the conclusion that he left out things he had done more often than he even seemed to claim to have done things he had not. Philadelphia during the great years of the Junto (1727 to 1757) was a town of remarkable intellectual activity, and its history has not yet been truly written as I hope it will some day be. But no matter what claims may be made for other men, Franklin emerges as the chief among them, the energizing, galvanizing source of two-thirds of the town's important enterprises.

When I am told, as I occasionally am, that I make Franklin out as larger than life, I can only answer that Franklin must have been what he was, because nobody could have invented such a figure. Stranger things happen in fact than in fiction. Nature is richer in invention than men are. The great characters in fiction are almost always heroes who have each of them some ruling passion, with enough human weaknesses to give him a reasonable credibility. Romantic creation is most likely to be exaggeration along a few lines. But the more you study Franklin the more lines you find running out from him in all directions, and the more facts that no poet—however romantic and exaggerative—would ever have thought of inventing. The wonder of Franklin is the facts that are true about him. The more exact the research into his character, the more surprising the adventure.

Too much emphasis has been laid, I think, on his simple practical ingenuities. You go into a grocer's and see the clerk taking objects down from high shelves with a device based on the "long arm" which Franklin invented to get at his books. Most of us have in our kitchens a combination chair-and-stepladder which Franklin seems to have devised; and in our fireplaces a draft such as he had made for his fireplace in Craven Street. He has been credited with the invention of the rocking chair, which I believe he did not invent, though in his last years he had one which automatically fanned him when he rocked. But the ingenuity that went into these gadgets is less notable than the fundamental ideas behind Franklin's principal inventions.

If, for example, we read his remarkable pamphlet about what he called the Pennsylvanian fireplace, later known as the Franklin stove, we should not be too much taken by the salesmanlike adroitness of his arguments. It is true he was adroit, particularly in his claims that the use of the stove would be beneficial to the health, complexion, and beauty of women. Such arguments were as likely to be effective in 1744 as they would be in 1939. But Franklin believed they were sound, as they were. He had not designed his stove purely for economy in fuel. He had strongly in mind the great importance of proper ventilation, and its value for health. He even took into account an esthetic element. His stove, unlike the Dutch and German stoves of the time, allowed people to see the fire, "which is in itself a pleasant thing." In 1744 Franklin had been using his stove, he said, for "the four winters past," which takes the invention back to 1740. But he put off writing about it till he had announced the organization of the American Philosophical Society. His pamphlet was in effect his first contribution to the work of this new league of scientists. As if to give his little treatise the dignity of learning, he accompanied it with notes from various impressive sources, one of them in Latin.

Certainly the lightning rod was not a gadget. The experiment which Franklin proposed, to prove whether electricity and lightning were identical, and his own separate demonstration with the kite, must be ranked with the most fundamental as well as the most striking experiments in scientific history. The story of the kite is now so old and so familiar that it has come to seem a pleasant legend, not much more real to us than the customary pictures of the scene, which show Franklin's son as a little boy when in fact he was twenty-one or so and as tall as his father. The experiment, because it solved a mystery, has so deprived lightning of its terror that it no longer overawes men. Franklin, drawing the lightning from the skies, removed it from the dread region of mythology. Kant was not speaking for picturesque effect when he said Franklin was a new Prometheus who had stolen fire from heaven. The expression meant, literally, that Franklin had made men equals of the gods and therefore free of an ancient slavish dread. Nobody in 1752 felt that the

kite story was a quaint little incident. It was something immense, and it gave Franklin the reputation of a wizard, not too much unlike Merlin or Roger Bacon—or, in our day, Einstein.

I have, I suppose, already made it clear that I do not agree with those almost unanimous modern commentators on Franklin who think of him as primarily an ingenious inventor. I will go further and say that I think his fundamental conjectures are more important than his inventions. He said of himself: "I own I have too strong a penchant to the building of hypotheses; they indulge my natural indolence." But these hypotheses were as truly original as anything he ever invented. What he called his "conjectures and suppositions" about electricity make up the Principia of the science. Nor did he confine himself to one branch of science, or to science as a whole. His notes in 1743 on the origin of northeast storms were the first step toward a scientific meteorology. In his "Observations Concerning the Increase of Mankind, Peopling of Countries, etc.," written in 1751, he not only anticipated Malthus, who acknowledged his debt to Franklin, but also forecast the theory of the American frontier later associated with the name of Frederick Jackson Turner. In 1754 Franklin in his famous letters to Governor Shirley of Massachusetts set down his far-sighted plan for equal justice to the various parts of the British Empire, and summed up almost all the American arguments of the Revolution. In 1762 he wrote the earliest piece of scientific musical criticism, and in 1768 said as much as has been said since about the need of reform in English spelling. He was the first scholar who studied the Gulf Stream (1769) and had some understanding of it and the possible use of it by navigators; and among the first who insisted that the common cold is more likely to come from contagion than from exposure. As he put it in one of the notes he made 1773 for a paper he intended to write: "Think they get cold by coming *out* of such hot rooms; they get them by being *in*."

Though from 1773 to 1783 Franklin was so much absorbed in politics he had little time for general ideas, he had hardly signed the final treaty of peace with England when his mind was alert with bold conjectures again. Having seen the first

ascent of human passengers in a free balloon, in Paris in November 1783, Franklin at once—and apparently alone among his contemporaries—foresaw the possibility of aerial warfare. This discovery, he wrote in December, might "give a new turn to human affairs. Convincing sovereigns of the folly of wars may perhaps be one effect of it; since it will be impossible for the most potent of them to guard his dominions. Five thousand balloons, capable of raising two men each, would not cost more than five ships of the line; and where is the prince who can afford so to cover his country with troops for its defence as that ten thousand men descending from the clouds might not in many places do an infinite deal of mischief before a force could be brought together to repel them?" As I speak armed forces in Europe are each hesitating to attack for fear of the very consequences Franklin foresaw at a glance.

Franklin's own age knew him as philosopher and sage, statesman and wit, and while delighting in his charm and grace thought of him as always a figure of weight and dignity. It remained for another century to take affectionate familiarities with him, and call him Ben Franklin—which only his immediate family ever did—as the century called other heroes Andy Jackson and Abe Lincoln. The homely anecdotes of Franklin's Autobiography gave him a homespun reputation which does him less than justice. There was little that was shirt-sleeved in his science or politics or diplomacy. His manners, as I have written elsewhere, "were as urbane and expert as his prose." His economical maxims give a wrong impression of his character, which was generous and at times lavish. "Avarice and happiness never saw each other," he wrote as Poor Richard. "How then should they become acquainted?" So far from making a great virtue of the cunning which has been often ascribed to him, Franklin as Poor Richard said that "Cunning proceeds from want of capacity" —meaning that truth was better. "Dr. Franklin," Henry Laurens wrote when the British ministers were warily looking for some one to treat with Franklin for peace in 1782, "knows very well how to manage a cunning man; but when the Doctor converses or treats with a man of candor there is no man more candid than himself."

Though Franklin was an excellent and successful business man, he retired from active business at forty-two and spent forty-two years more in the service of the public. He might have made a fortune if he had patented his stove or his lightning-rod. He refused to patent anything which he thought might be of benefit to mankind. As he did not hungrily gather wealth, so he did not cautiously guard his comfort or safety. It must never be forgotten that in his seventieth year Franklin might with decency have done what his more conservative son advised him to do: that is, retire from active affairs and let younger men settle the conflict between England and America. Instead Franklin, at the risk of peace and even of his neck, took his stand with the revolutionaries. Life with him began all over again at seventy. The older the bolder.

I shall take the liberty of reading the final words of my "Benjamin Franklin," in which I have done my best to reduce his qualities to their essence. "Franklin was not one of those men who owe their greatness merely to the opportunities of their times. In any age, in any place, Franklin would have been great. Mind and will, talent and art, strength and ease, wit and grace met in him as if nature had been lavish and happy when he was shaped. Nothing seems to have been left out except a passionate desire, as in most men of genius, to be all ruler, all soldier, all saint, all poet, all scholar, all some one gift or merit or success. Franklin's powers were from first to last in a flexible equilibrium. Even his genius could not specialize him. He moved through his world in a humorous mastery of it. Kind as he was, there was perhaps a little contempt in his lack of exigency. He could not put so high a value as single-minded men put on the things they give their lives for. Possessions were not worth that much, nor achievements. Comfortable as Franklin's possessions and numerous as his achievements were, they were less than he was. Whoever learns about his deeds remembers longest the man who did them. And sometimes, with his marvelous range, in spite of his personal tang, he seems to have been more than any single man: a harmonious human multitude."

CONTROL PROBLEMS.

BY

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(Continued from November issue.)

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PART III.

Stability of Controlled Systems.

18. *General Considerations.*

The question of stability of a controlled process is of a major practical importance and can be considered from several angles, according to what is meant by stability in each case. One can consider several kinds of stability viz.:

(1) Inherent stability of system (A) to be controlled.
(2) Stability of the controlled system ($A + S$) or "controlled stability."
(3) Stability of the control system (S) per se.

In a given practical problem all three forms of stability (or instability) generally appear simultaneously and it is difficult to discriminate between the cause and the effect in a closed cycle of the control process. It is necessary, therefore, to analyze these conditions separately.

As far as inherent stability of the system (A) is concerned, it can be ascertained in each individual case by a priori considerations in a relatively simple manner. Only in a few control problems system (A) is inherently stable. To this case belongs, for instance, the problem of anti-rolling stabilization of ships; in fact, a ship is an inherently stable system, having the upright position as that of a stable equilibrium; the problem of control in this case is not so much to *provide* an inherent stability but rather *to help* it by counteracting the cumulative action of waves which might otherwise produce considerable departures (large rolls) from this stable upright position.

If one considers, on the other hand, the problem of automatic steering—the situation is quite different. A ship, following initially a certain course with rudder amidships, not only does not have any tendency to maintain this course but inevitably departs from it like an inverted pendulum; the stability is thus essentially negative in this case (5, 9).

Finally, in a great majority of control problems, there is no inherent stability at all and one has to provide an artificial "set point" around which the control is supposed to operate.

Only in the case of an inherent instability of system (A) special conditions must be fulfilled in order to convert this inherent instability into stability, by making the gradient of the restoring action due to the control, greater than the gradient of the inherent instability couple (Section 27 below).

The question of inherent stability (or instability) is thus of a relatively small interest for the control problem proper; it relates, in fact, to the system (A) itself.

The second question—stability of the controlled system ($A + S$) is more important. In fact, it happens frequently that the operation of a control system produces phenomena of *dynamical instability*, or "floating" which the original system (A) does not exhibit. Under certain conditions, the controlled system ($A + S$) may even exhibit a tendency to

deviate from the equilibrium point entirely. Investigation of such conditions forms the principal object of this Part III.

Finally the control system per se—may exhibit a local instability; this form of instability is less alarming than the case (2) as long as such local instability does not affect the controlled system $(A + S)$ as a whole but manifests itself only in the control mechanism itself. In practice these "parasitic" oscillations or "hunt" in a control system, merely reduce its efficiency without causing a serious reaction on the controlled process, unless this local instability in the control mechanism becomes excessive.

The usual definition of stability is generally as follows: *The system $(A + S)$ is said to be stable if, being given a small departure from its equilibrium position by means of an initial disturbance, it has a spontaneous tendency to come back to this position of equilibrium in the course of time, after the disappearance of the disturbance.* For linear systems acted upon by linear controls—which forms the object of this investigation—this is equivalent to a statement that the real parts of complex roots of the characteristic equation, corresponding to differential equation $[R]\theta = 0$ of the controlled process, *must be all negative.* This definition absorbs the case of real roots as well. In such a case, the general solution of differential equation of the controlled process is represented by a linear assembly of terms of the form t^k or $t^k \cos \omega t$ (k being a positive integer or zero), each having $B_i e^{-\alpha_i t}$ as a factor; these terms, therefore, die out for $t \to \infty$.

Such roots are frequently referred to, in connection with problems of stability, as "stable roots."

19. *Hurwitz's Theorem.*

Routh (10), Hurwitz (11) and Liapounoff (12) have given different criteria for the existence of stable roots. For linear problems with which we are concerned here, Hurwitz's theorem gives a relatively simple procedure for ascertaining the existence of stable roots and can be applied to equations of any degree. For this reason, it is given below without proof.[1]

Hurwitz's theorem states: conditions necessary and sufficient for the existence of stable roots in an algebraic equation

[1] Proof of the Hurwitz's theorem can be found in references (11) and (13).

of the nth degree:

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

are: *all "Hurwitzian determinants" H_1, H_2, H_3, $\cdots$ must be positive.* By "Hurwitzian determinants" in this statement, are meant the subsequent minors of the following array of coefficients a_i

$$\begin{array}{|cc|cc|c} a_1 & a_0 & 0 & 0 & 0 \cdots \\ \hline a_3 & a_2 & a_1 & a_0 & 0 \cdots \\ \hline a_5 & a_4 & a_3 & a_2 & a_1 \cdots \\ \hline a_7 & a_6 & a_5 & a_4 & a_3 \cdots \\ \hline \end{array} \qquad (53)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

The following remarks are important in forming these Hurwitzian determinants H_i.

(1) The first coefficient a_0 must always be positive. If it is not positive, the equation must be multiplied by -1 which makes $a_0 > 0$ and the procedure must be applied only after this change of sign.

(2) If the equation is of the nth degree, all coefficients of higher indices, viz., a_{n+1}, $a_{n+2} \cdots$ should be replaced by zeros in the array (53).

(3) Hurwitz's theorem states that for stable roots, *all* Hurwitzian determinants must be positive. If at least one among them is not positive (i.e. $\leqslant 0$), one cannot affirm that the stability exists. More likely, unstable condition will set in. With these points in mind, the application of the Hurwitz's theorem for algebraic equations is as follows:

1. *Quadratic Equation:*

$$a_0x^2 + a_1x + a_2 = 0; \qquad (a_0 > 0).$$

Hurwitzian determinants:

$$H_1 = a_1 > 0$$
$$H_2 = \begin{vmatrix} a_1 a_0 \\ 0\ a_2 \end{vmatrix} = a_1a_2 > 0.$$

Since $a_1 > 0$ from H_1, one concludes from H_2 that $a_2 > 0$. The Hurwitzian conditions in this case are reduced to: $a_0 > 0$; $a_1 > 0$; $a_2 > 0$; that is, for stable roots all coefficients of the quadratic equation must be positive. This can be verified directly in this case.

2. *Cubic Equation:*

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0; \qquad (a_0 > 0).$$

Hurwitzian criteria are:

$$H_1 = a_1 > 0$$
$$H_2 = a_1a_2 - a_0a_3 > 0$$
$$H_3 = a_3(a_1a_2 - a_0a_3) > 0.$$

In view of H_2, the last criterion is simply $a_3 > 0$. Since a_0, a_1 and a_3 are positive, from the second criterion it follows that $a_2 > 0$. In other words, all four coefficients a_0, a_1, a_2, a_3 must be positive. The same rule (all coefficients must be positive) applies also to the quartic equation. This rule does not hold, however, for still higher degrees (5th, 6th, etc.), and the Hurwitzian criteria must be used directly in such cases.

3. *Quartic Equation:*

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0; \qquad (a_0 > 0).$$

Hurwitzian criteria:

$$H_1 = a_1 > 0$$
$$H_2 = a_1a_2 - a_0a_3 > 0$$
$$H_3 = a_3(a_1a_2 - a_0a_3) - a_1^2a_4 > 0$$
$$H_4 = a_4[a_3(a_1a_2 - a_0a_3) - a_1^2a_4] > 0.$$

In view of H_3, the last criterion is equivalent to: $a_4 > 0$. One could proceed in this manner for equations of still higher degree. The following general remark holds: the second (a_1) and the last (a_n) coefficients must be positive.

Note: for $n = 3, 4$ the rule,—all coefficients must be positive,—gives only a *necessary* condition for stable roots, whereas the Hurwitzian criteria give *both necessary and sufficient* conditions.

20. *Retarded Control Action.*

In the preceding analysis of effects of different control terms it was tacitly assumed that the corresponding control actions are instantaneously in phase with θ, $\dot{\theta}$, $\ddot{\theta}$ of motion of the resultant system $(A + S)$.

In reality, in view of time lags (electrical, mechanical, thermal, etc.), present in any control system the real control action lags behind its theoretical "non-retarded" value.

In other words, it has been assumed so far that the various "control terms," e.g.:

$$S_0^0 = a_0^0\theta; \qquad S_1^0 = a_1^0\dot{\theta}; \qquad S_2^0 = a_2^0\ddot{\theta} \cdots$$

are of the form:

$$S_0^0 = a_0^0\theta(t); \qquad S_1^0 = a_1^0\dot{\theta}(t) \cdots,$$

where $\theta(t)$, $\dot{\theta}(t)$, $\ddot{\theta}(t) \cdots$, *are functions of time t, considered at the instant t.*

In view of the time lag Δt (which for the sake of simplicity will be assumed the same for various control terms) these functions θ, $\dot{\theta}$, $\ddot{\theta}$ are *retarded functions* and relate to a past instant $t - \Delta t$ so that instead of $\theta(t)$, $\dot{\theta}(t)$, $\ddot{\theta}(t) \cdots$ one should consider the *retarded values* of these functions, i.e., $\theta(t - \Delta t)$, $\dot{\theta}(t - \Delta t) \cdots$ Since, for continuous control processes with which we are concerned here, the functions θ, $\dot{\theta}$, $\ddot{\theta} \cdots$ are also continuous, one can expand them in Taylor's expansion. This gives:

$$\bar{\theta}(t) = \theta(t - \Delta t) = \theta(t) - \Delta t\dot{\theta}(t) + \frac{\Delta t^2}{2!}\ddot{\theta}(t) - \cdots, \quad (54)^2$$

$$\bar{\dot{\theta}}(t) = \dot{\theta}(t - \Delta t) = \dot{\theta}(t) - \Delta t\ddot{\theta}(t) + \frac{\Delta t^2}{2!}\dddot{\theta}(t) - \cdots, \quad (55)$$

. . . . .

where the bars above θ, $\dot{\theta}$, $\ddot{\theta}$, i.e., $\bar{\theta}$, $\bar{\dot{\theta}}$, $\bar{\ddot{\theta}}$ designate the *retarded values* of these functions corresponding to the instant $t - \Delta t$. Since no further confusion is to be feared, one can drop, from now on, the functional notations and write θ, $\dot{\theta} \cdots$ instead of $\theta(t)$, $\dot{\theta}(t) \cdots$ etc. With these simplified notations, the effect of

[2] For the sake of simplicity of notations Δt^2, Δt^3, $\cdots$ in the following designate $(\Delta t)^2$, $(\Delta t)^3$, $\cdots$.

retarded control terms on the controlled process can be easily ascertained. Consider, as an example, the control term $a_1\dot{\theta}$ (the upper index in a_1^0 is dropped for the sake of simplicity). If the control is an *ideal* one, i.e., non retarded, its effect on the system (A) leads to the differential equation:

$$A_2\ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0\theta = 0. \tag{56}$$

This is the differential equation of the controlled pendulum investigated in section 17. For a retarded control $a_1\dot{\bar{\theta}}$ one has to substitute for $\bar{\theta}$ its expression (55) which gives instead of equation (56) the following "retarded differential equation" of the controlled process:

$$a_1\left[\cdots(-1)^n\frac{\Delta t^n}{n!}\theta^{(n+1)} + \cdots - \frac{\Delta t^3}{3!}\theta^{IV} + \frac{\Delta t^2}{2!}\dddot{\theta}\right] + (A_2 - a_1\Delta t)\ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0\theta = 0. \tag{57}$$

It is seen that the differential equation (57) of the controlled process is now of an infinitely high order, while still being linear and with constant coefficients, assuming that Δt remains constant.

This type of equation is designated sometimes as *hystero-differential* equation;[3] this designation will be used below. There exists, so far, no general theory, of these equations, but from the fact that equation (57) is linear and with constant coefficients one can formulate the following fundamental properties of its integrals.

(1) There exists an infinite number of frequencies of possible modes of oscillations in view of an infinite number of roots of its characteristic equation (section 24 below).

(2) The principle of superposition of solutions still holds.

(3) For some of the frequencies, condition of self-excitation may be fulfilled; this means that under certain conditions, there may appear complex roots with real positive parts. When these conditions are fulfilled, the controlled system $(A + S)$ begins to "hunt" or execute parasitic oscillations with frequencies which generally are higher than the free frequency of the system (A). In view of the fact that this

[3] This designation was introduced by L. Silberstein (14).

particular phenomenon of "hunt" has been investigated elsewhere (15) this subject is not treated here.

The problem in general is a very complicated one, and no attempt will be made to go into its detailed study here. It is possible, however, to see what happens in the two limiting cases; (*a*) when the time lag is small enough in comparison with one of the component periods of the process and, (*b*) when this lag is of the same order as that period.

The last case has already been studied (15), and a conclusion thus obtained shows the existence of rapid oscillations or "hunt" in the control system. In fact, practically all control systems hunt, more or less, and the elimination of the hunt constitutes generally a major problem of design.

In the first case (*a*) the convergence of Taylor's expansion is relatively good and it is sufficient to consider only a finite number of terms which permits, thus, gaining an insight as to the limit process involved. If the third and the fourth order terms are retained, one can expect the appearance of a new frequency of free oscillation. Unfortunately, there is no simple way of determining this new frequency because of a practical impossibility of solving a quartic equation algebraically.[4] A. Blondel (16) has indicated approximate solutions sufficient for practical purposes but this still is a relatively complicated procedure and will not be considered here. More important is the question of stability of such retarded control systems and this subject will be analyzed now.

21. *Stability of Retarded Control Systems.*

It was shown that the effect of a retarded control results in the hystero-differential equation (57). The difficulties in connection with the latter are considerable. Two ways are open in this connection.

(a) A rigorous procedure consists in transforming the characteristic equation of an infinitely high degree corresponding to the hystero-differential equation by a convergence process which results in a transcendental equation possessing an infinity of roots. This problem is treated in sections 23 and 24 below.

[4] This can be done, of course, numerically, if the values of different coefficients are known.

(b) An approximate step by step solution based on the fact that the Taylor's expansions encountered in practical control problems converge rapidly if the time lags are small enough with respect to the fundamental period of oscillation. It is possible then to form an idea about the stability conditions of a controlled process by a method of successive approximations and gain thus a qualitative insight as to the limit of convergence of the problem. This latter course is followed in this section.

Consider the hystero-differential equation (57) in which only the first order of Δt is retained and the higher orders $\left(\frac{\Delta t^2}{2!}, \frac{\Delta t^3}{3!} \cdots\right)$ are neglected as being small. One has:

$$(A_2 - a_1\Delta t)\ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0\theta = 0. \tag{58}$$

It is seen that the retarded action, as far as its manifestation to the first order is concerned, reduces somewhat the coefficient of inertia A_2 of the system (A) to be controlled. This is generally a very small effect if Δt is small and the inertia factor A_2 of the system (A) is considerable in comparison with A_1, and A_0 which applies generally to practical problems of stabilization, automatic steering, etc. If one takes a step further and retains Δt and $\frac{\Delta t^2}{2!}$ but neglects still higher orders $\left(\frac{\Delta t^3}{3!}, \frac{\Delta t^4}{4!} \cdots\right)$, one has:

$$\frac{\Delta t^2}{2!}\dddot{\theta} + (A_2 - a_1\Delta t)\ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0\theta = 0. \tag{59}$$

The application of the Hurwitz's criteria still indicates the existence of stable roots and, hence, of stability.

If we pass now to the third order, we have:

$$a_1\left[-\frac{\Delta t^3}{3!}\theta^{IV} + \frac{\Delta t^2}{2!}\dddot{\theta}\right] + (A_2 - a_1\Delta t)\ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0\theta = 0. \tag{60}$$

By virtue of remark (1) section (19), this equation should be multiplied by -1 i.e., its signs changed and one observes that the Hurwitzian criteria are not fulfilled any more.

Beginning with this approximation and by taking more and more terms in the hystero-differential equation, one finds that the Hurwitzian criteria of stability are not fulfilled. It is seen, thus, that the effect of instability is felt only to a relatively high (the fourth, in this case) order; in other words, unless the time lag is appreciable, its manifestation on stability is relatively small. One can also see this from a direct consideration of equation (60) for example. In fact, the characteristic equation corresponding to equation (58) for $\Delta t = 0$ is:

$$A_2 r^2 + (A_1 + a_1) r + A_0 = 0$$

and this equation has stable roots. On the other hand, for the time lag retained to the fourth order (equation 60), the characteristic equation is:

$$\frac{\Delta t^3}{3!} r^4 - \frac{\Delta t^2}{2!} r^3 - (A_2 - a_1 \Delta t) r^2 - (A_1 + a_1) r - A_0 = 0.$$

Hurwitzian criteria of stability for this last equation are not fulfilled as is easy to ascertain, using the procedure of Section 19, that this instability is due exclusively to the retarded control system ($\bar{S}$) and not to the original non-retarded system ($A + S$). Thus, for a stable system (A) and for a stable non-retarded control (S), the resultant system ($A + S$) is stable; for a retarded control ($\bar{S}$), however small the time lag is—the resultant system becomes theoretically unstable.

This apparent paradox should be understood, however, in the following sense. The Hurwitz's theorem permits to ascertain only whether the system is stable or not in the immediate vicinity of the equilibrium position but does not give any criterion as to "how much unstable" is the system, if it is unstable. It follows, therefore, that it may happen that a theoretically unstable system by falling off from the position of unstable equilibrium may reach such a region in which the linear approximation used in the setting up of the differential equation ceases to be applicable. Non-linear phenomena due to energy dissipation, friction, etc., may limit its further departure and the process may even reverse; the system will be thrown back to the original unstable equilibrium point; after this it will overshoot this equilibrium

point and will continue to progress to the other side, etc. In this manner, a parasitic oscillation or "hunt" will be started. As a matter of fact, the hunt is a manifestation of an essentially unstable condition of the system because it can continue indefinitely, so that the system will never come to the equilibrium point in this manner.

A retarded control action, thus, always results in an instability however small time lag is. In reality, for small time lags, the non-linear factors such as Coulomb friction, for instance, may reduce or even eliminate this effect of instability—this does not mean, however, that such a residual *asymptotic instability* is not present, at least to a small degree. Conversely, if conditions are such that the system is fairly linear in the vicinity of equilibrium (e.g. small Coulomb friction) and time lag is relatively large—the presence of several frequencies in these parasitic oscillations can be observed which, in fact, is in conformity with the nature of the hystero-differential equation having an infinite spectrum of possible frequencies. These interesting, but little explored, phenomena can be particularly well observed in control systems with large time lags and great "intensity coefficients" (a_1 in equation 57); Taylor's expansion in such a case has a relatively poor convergence and the difference between the hystero-differential equation actually governing the process and the differential equation (51–1) applicable for vanishingly small time lags—becomes appreciable.

The above discussion was carried out in connection with the control term $a_1\theta$ and it was found that the effect of asymptotic instability begins to be felt only to a relatively high order. If one considers control terms of a higher order, e.g., $a_2\ddot{\theta}$, $a_3\dddot{\theta}$ $\cdots$ one finds, following the same line of argument, that this effect is considerably more pronounced i.e., occurs already for orders lower than the fourth order, previously established. In fact, experiments show that the instability sets in more easily if higher order control terms are used.

22. *Anticipatory Control.*

From these considerations it follows that a great deal of complexity characterizing a retarded control process, such as instability with incident parasitic oscillations, arises from the

fact that such a process in reality, is governed by a hystero-differential equation and not by an ordinary differential equation of a finite order, which holds only for zero time lag as was tacitly assumed throughout the first two parts of this paper.

Since this complication arises from the fact that the real control actions are always *retarded actions*, the practical problem which presents itself in this manner, is to attempt to obtain a method by which the retarded quantities in a control could be converted into non-retarded ones with the incidental reduction of the mathematical problem from the realm of a hystero-differential equation back to an ordinary differential equation of a finite order—in which case the control process is simple. The problem of this kind is designated sometimes, although perhaps not quite properly, as *the problem of anticipatory control.*[5]

In order to specify this problem, one has to consider again a few terms of the hystero-differential equation (57), for example the differential equation (60) retaining the fourth order. In this manner, one retains an additional pair of roots, which in a great majority of cases is a pair of complex conjugate roots with positive real parts, since the Hurwitzian conditions are not fulfilled in this case. This results in a parasitic oscillation in the controlled process which, in fact, is generally observed.

Putting for the sake of abbreviation $m_4 = a_1 \frac{\Delta t^3}{3!}$ and $m_3 = a_1 \frac{\Delta t^2}{2!}$ in equation (60) and neglecting $a_1 \Delta t$ in comparison with A_2 which is generally justified in practice, this equation becomes:

$$- m_4 \theta^{IV} + m_3 \ddot{\theta} + A_2 \ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0 \theta = 0. \quad (61)$$

The differential equation (61) represents an approximation of the hystero-differential equation (57) to the fourth order with a view to evidencing at least one pair of new conjugate roots in the unstable process due to time lag.

Assume now that we introduce two additional control terms, $+ \mu_4 \theta^{IV} - \mu_3 \ddot{\theta}$,—the anticipatory control terms—this

[5] A practical example of an anticipatory control can be found in reference (17).

gives:

$$(\mu_4 - m_4)\theta^{IV} + (-\mu_3 + m_3)\ddot{\theta} + A_2\ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0\theta = 0. \quad (62)$$

If one adjusts the intensity coefficients μ_4 and μ_3 of this anticipatory control, so as to have $\mu_4 = m_4$ and $\mu_3 = m_3$, the differential equation (62), if the higher order terms are negligible, becomes:

$$A_2\ddot{\theta} + (A_1 + a_1)\dot{\theta} + A_0\theta = 0. \quad (51\text{-}1)$$

This, however, is the differential equation of an ideal, non-retarded control considered in detail in Part I.

Theoretically, this compensation of m_i by μ_i should be carried out for still higher order ($i = 5, 6 \cdots$). However, in view of a rapid convergence of the Taylor's expansion, in practice, it is already sufficient to carry out this compensation up to the fourth order which eliminates the first parasitic frequency which generally is the most important one.

In practice, this compensation requires certain precautions. In fact, coefficients m_3 and m_4 contain a_1 as a factor. If, therefore, the coefficient a_1 of the basic control is varied, the compensating terms μ_3 and μ_4 of the anticipatory control should be varied in proportion to a_1. This can be accomplished by obtaining the anticipatory control from the basic control through a number of differentiating circuits which automatically establishes this proportionality between μ's and m's. It remains, then, to adjust individually μ_4 and μ_3 until the behavior of control system becomes quiet and the parasitic frequency disappears. The problem of anticipatory control is difficult, however, because in this particular example, one has to go to three subsequent differentiations, which requires special precautions in differentiating circuits. Once, however, the circuits have been adjusted—this adjustment generally remains fixed and the controlled system behaves in accordance with the differential equation (51-1) rather than with the hystero-differential equation (57) with all complications which it entails.

For an overcompensated anticipatory control, i.e., when $\mu_4 - m_4 \gtrless 0$ and $(-\mu_3 + m_3) \gtrless 0$, the situation may become considerably worse and violent parasitic oscillations may occur in the free oscillation of the controlled process.

23. *Extension of Operational Notations to Retarded Controls.*

In Chapters I and II dealing with the behavior of ideal control systems, use was made of operational notations. It is necessary now to generalize these notations for retarded quantities characterizing the real control systems.

The principal complication which appears in this connection is in the fact that the controlled process is governed in this case by a hystero-differential equation (57) and not by an ordinary differential equation of a finite order as in the case of ideal control systems considered in Parts I and II above.

From the fact, however, that the hystero-differential equation is still a linear differential equation with constant coefficients, but only of an infinitely high order, one can expect that the application of linear operators is still permissible if one finds a limit—process by which the application of linear operators of a finite order could be extended up to an infinitely high order. Such a limit-process is, in fact, available in view of the convergence of Taylor's expansion characterizing the retarded actions, as will be shown now.

Consider, as an example, the same control term $S_1^0 = a_1\theta$ (Section 21). For a retarded action, we have to write: $\bar{S}_1^0 = a_1\bar{\theta}$ where $\bar{S}_1^0$, or more correctly, $[\bar{S}_1^0]$ is a *retarded control operator.*

An explicit expression for such a retarded operator can be obtained in the following manner:

Consider again equation (55) for a retarded function $\bar{\theta}$:

$$\bar{\theta} = \dot{\theta} - \Delta t\cdot\ddot{\theta} + \frac{\Delta t^2}{2!}\dddot{\theta} - \frac{\Delta t^3}{3!}\theta^{IV} + \cdots. \tag{55}$$

The right hand of this differential form can clearly be put in the form of an operator, viz.:

$$\left[\delta - \Delta t\cdot\delta^2 + \frac{\Delta t^2}{2!}\delta^3 - \frac{\Delta t^3}{3!}\delta^4 + \cdots\right], \tag{63}$$

operating on θ. Taking δ as factor, this operator can be written also in the form:

$$\left[\delta\left(1 - \Delta t\cdot\delta + \frac{\Delta t^2}{2!}\delta^2 - \frac{\Delta t^3}{3!}\delta^3 + \cdots\right)\right] = [\delta e^{-\Delta t\delta}]. \tag{64}$$

On a purely formal basis, we can now write the left hand of equation (55) in operational form as: $\bar{\theta} = [\bar{\delta}]\theta$ where $[\bar{\delta}]$ is a *retarded operator* $[\delta]$. Whence

$$[\bar{\delta}] = [\delta e^{-\Delta t \cdot \delta}],^{6} \tag{65}$$

which gives the explicit value of $[\bar{\delta}]$—the retarded control operator. The resultant operator $[R]$ of the system $(A + S)$ is then:

$$[R] = [A_2\delta^2 + (A_1 + a_1 e^{-\Delta t\delta})\delta + A_0]. \tag{66}$$

Operator (66) in spite of its non-linear appearance, behaves in some respects as a linear operator, because the exponential $e^{-\Delta t\delta}$ is the limit of the infinite series of linear operators and contains already the passage to the limit in which we are interested.

It is useful to give an application of operator (66) to a concrete example. In this connection, one is to be guided by the fact that the hystero-differential equation being a linear differential equation with constant coefficients of an infinitely high order has an infinite number of roots. A particularly interesting case is when the roots of this infinite spectrum are distinct and purely imaginary; such a case indicates the existence of a steady state of parasitic oscillations or "hunt" in the system. If we postulate, therefore, that these oscillations are, indeed, present, clearly $\theta_j = \theta_{0j}e^{i\omega_j t}$ where j is the order of a parasitic oscillation of frequency ω_j and amplitude θ_{0j}. One can drop the index j for the sake of simplicity, keeping in mind, however that, theoretically, there is an infinity of these oscillations forming a discrete spectrum.

In such a case $\dot{\theta} = \delta\theta = i\omega\theta$; $\ddot{\theta} = \delta^2\theta = (i\omega)^2\theta$ etc. We are, thus, led to identify the symbol δ (the differentiation) with $i\omega$ in this particular case.

One has to remember, however, that in so doing, it is assumed that operator (66) *has already operated on the particular solution* $\theta = \theta_0 e^{i\omega t}$.

As the result of this, the substitution $\delta = i\omega$ in the resultant operator $[R]$ *gives directly the characteristic equation* of the

[6] This differential operator was known to Lagrange who considered it as a kind of a "hypercomplex unit" (N. Wiener, "The Operational Calculus," *Math. Annalen*, 94, 1925).

hystero-differential equation, viz.:

$$A_2(i\omega)^2 + (A_1 + a_1 e^{-i\omega\Delta t})i\omega + A_0 = 0. \tag{67}$$

The discrete spectrum of frequencies ω characterizing the controlled process is given then by the roots of this characteristic equation. Putting $\omega\Delta t = \varphi$ and separating real and imaginary terms, one has, after simple calculations:

$$(A_0 + a_1\omega \operatorname{Sin}\varphi - A_2\omega^2) + i(A_1\omega + a_1\omega \operatorname{Cos}\varphi) = 0,$$

whence two equations:

$$a_1\omega \operatorname{Sin}\varphi = A_2\omega^2 - A_0$$
$$a_1\omega \operatorname{Cos}\varphi = -A_1\omega.$$

Dividing them out, one has:

$$\operatorname{Tan}\varphi = \frac{A_0 - A_2\omega^2}{A_1\omega}. \tag{68}$$

24. *Parasitic Frequencies Spectrum of a Retarded Control.*

We are now in a position to explore the frequency spectrum of parasitic oscillations which may arise in a controlled system with a retarded control.

Although this problem reminds closely a similar problem already investigated from a somewhat different angle (15) a brief review of the procedure is still of interest here, because this time we reach a similar conclusion from the definition of the retarded control operator $[a_1\delta e^{-\Delta t\delta}]$.

Transcendental equation (68) is in fact equivalent to equation (67). It can be solved either by infinite series or graphically; the last mentioned method is applied below.

Since $\phi = \omega\Delta t$, the graphical solution is as follows (Fig. 2). Let the frequencies ω be plotted on the abscissa axis; on ordinates are plotted two functions, Tan ϕ and $\frac{A_0 - A_2\omega^2}{A_1\omega}$. The first function undergoes discontinuities for $\phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots \frac{(2n+1)\pi}{2} \cdots$, that is for $\omega_1 = \frac{\pi}{2\Delta t}$, $\omega_3 = \frac{3\pi}{2\Delta t} \cdots$, $\omega_{2n+1} = \frac{(2n+1)\pi}{2\Delta t} \cdots$, and goes through zeros for $\phi = 0, \pi,$

$2\pi \cdots$ that is for $\omega_0 = 0$; $\omega_2 = \frac{2\pi}{2\Delta t}$; $\cdots$ $\omega_{2n} = \frac{2n\pi}{2\Delta t}$ $\cdots$ as shown.

The second function $\frac{A_0 - A_2\omega^2}{A_1\omega} = \frac{A_0}{A_1\omega} - \frac{A_2}{A_1}\omega$ is equal to the algebraic sum of ordinates of a hyperbola $y_I = \frac{A_0}{A_1}\frac{1}{\omega}$ referred to coordinate axes as asymptotes plus those of

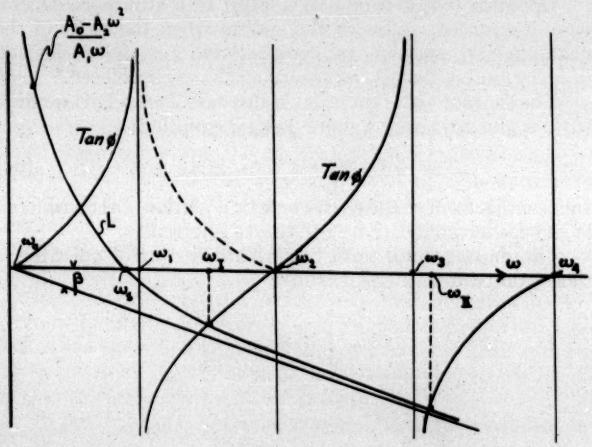

FIG. 2.

a straight line $y_{II} = -\frac{A_2}{A_1}\omega$, having angular coefficient $\text{Tan}\,\beta = -\frac{A_2}{A_1}$. Condition (68) of persistence of undamped oscillations of the form $\theta = \theta_0 e^{i\omega t}$ in the system indicates that such a state is possible only for such discrete frequencies at which both curves for Tan φ and for $\frac{A_0 - A_2\omega^2}{A_1\omega}$ (curve L) intersect each other. Furthermore, it is generally observed that the parasitic oscillations have frequencies higher than

the synchronous frequency ω_S of system (A) at which and the curve L intersects the abscissa axis. This restricts the parasitic frequencies ω_I, ω_{II} etc., to the right of the point ω_S of the synchronous frequency.

There are additional considerations governing these complicated phenomena inherent in the nature of the hysterodifferential equation by which a retarded control process manifests itself; for these details, the reference is made to the earlier publication (15).

Operator (66) can be also applied to a still more general case of transient states of the system when the roots of the characteristic equation are complex viz., $z = \alpha + i\omega$ instead of being purely imaginary $z = i\omega$ as above considered.

The characteristic equation in this case, instead of equation (67), is given then by a more general equation:

$$A_2 z^2 + (A_1 + a_1 e^{-z\Delta t})z + A_0 = 0. \tag{69}$$

In fact, this form of the characteristic equation was considered in the earlier publication (15), above referred to.

The most general form of a retarded control operator is then (compare with equation 9):

$$[\delta] = \left[\sum_{-\lambda}^{+\mu} a_i \delta^i e^{-i\Delta t} \right]. \tag{70}$$

PART IV.

Non Linear Control Problems.

25. *Functional Control Operators.*

Except in a few special applications, this branch of control problems is, practically, non existent at present. The analytical difficulties in this case are far more considerable than in linear problems studied so far. Nevertheless, the potential possibilities of this case are sufficient to warrant, at least, a very superficial review of this subject (21).

One may mention on this occasion the adage: "Nature is not afraid of analytical difficulties." In other words, a non linear control mechanism capable of solving a particular control problem, may be still of interest, in spite of the fact

that the mathematicians do not know as yet how to solve non linear differential equations in general.

Furthermore, it is generally a very difficult matter to be able to ascertain, in a given problem, to what extent this particular problem can be considered as linear or as non linear and within which limits?

As a matter of fact, practically all physical problems are non linear; they are generally idealized by linear approximations either within a certain range of variation of their variables, or under certain simplified assumptions.

Thus, for instance, the method of Small Motions of Dynamics idealizes non linear problems by linear approximations if the motions remain *small* in the vicinity of the equilibrium point which permits retaining only linear terms in Taylor's expansions of different functions involved.

If there is friction, the customary idealization consists to assume that this friction is proportional to velocity which, generally, is not strictly true; for example, the presence of a discontinuous or "Coulomb friction" vitiates frequently the results of such an idealization, if the relative importance of this particular type of friction is not negligible.

In electrical control problems conditions are similar; circuits involving electron tubes, for example, can be considered as linear only approximately when these tubes operate on substantially rectilinear parts of their characteristics; if not, problems become non linear ones.

In problems of a non linear control, it is supposed that by its very nature, the control is non linear even for a very small (theoretically, infinitely small) range of variation of variables determining the process. This transfers the problem into an entirely different plane. In fact, in linear problems the combination of the control operator $[S]$ with the operator $[A]$ of the system (A) is a linear one and this, as was shown, either leads to a virtual modification of coefficients of the original differential equation for (A) without any change in its form (fundamental control), or adds a few linear terms to the original differential equation, which raises the order of the latter while still maintaining the linearity of the final differential equation. With non linear controls the changes are far more radical; in fact, a control of this type actually

transforms one functional relation into another one. This corresponds to functional transformations of the type encountered either in the theory of integral equations, or functional equations in general, in which one function is transformed into another by means of a *functional operator.*

It is not proposed to enter into the theory of these questions in this paper, particularly in view of the fact that the use of such non linear controls is yet very limited.

In Applied Mechanics, particularly in kinematics of mechanisms, such functional transformations are used extensively in connection with cams, lever systems, etc., transforming one type of motion into a motion of an entirely different kind. An ordinary slide rule is a typical example of a functional transformation of this kind; in this instance, a logarithmic scale permits transforming addition or subtraction of rectilinear segments into an equivalent multiplication and division of corresponding numbers.

As other examples, one can cite the well known transformations of different scales (e.g. quadratic, cubic, etc.) into a linear one. This problem is frequently encountered in connection with measuring instruments.

An example given below (18) will help to understand the nature of these functional transformations in connection with problems of a non linear control.

26. *Method of Integrating Contour.*

Consider a flat coil ABC Fig. 3 consisting of two rectilinear sides $AC = X$ and $BC = Y$ and a curvilinear side AB moved along the abscissa axis into the shaded area representing a sharply defined region of an alternating magnetic flux at right angles to the plane of Fig. 3. The flat coil ABC consists of a few turns of a thin wire wound around the edge of the profile ABC cut of a plastic non-magnetic and non-conducting material; a and b are the terminals of the coil.

It is assumed that the bounding curve AB is a certain known analytical function—the generating function, $y = f(x)$ of abscissa x.

It is clear, that the flux linkages through the contour of the coil ABC, and, hence, the amplitude E of the electromotive force induced in the coil between its terminals a and b, are

proportional to the area k of the coil within the flux, indicated by a double shading on Fig. 3.

Thus, $E = \mu' k$ where μ' is a constant factor of proportionality depending on the number of turns of the coil and flux density. Clearly,

$$k = k(x) = \int_0^x f(x)dx;$$

by choosing different generating functions $f(x)$ bounding the contour ABC different types of functions $k(x)$ can be obtained.

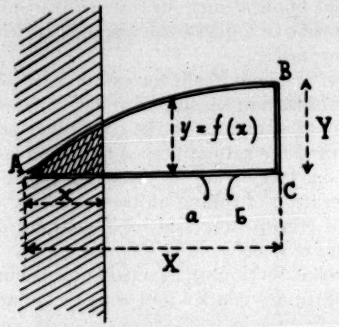

FIG. 3.

A few simple examples may be useful in illustrating the application of this method.

(1) *Triangular Contour.*

In this case is: $f(x) = \lambda x$, where λ is the angular coefficient ($\lambda = \text{Tan } \beta$) of the straight line AB; whence

$$k(x) = \int_0^x f(x)dx = \frac{\lambda}{2} x^2.$$

Thus, the amplitude of the electromotive force induced between a and b will be of the form:

$$E = \mu' k(x) = \mu x^2, \qquad (71)$$

where $\mu = \frac{\lambda}{2} \mu'$.

E can obviously represent the control quantity, e.g., the control voltage. If the displacement x of the contour represents a certain linear measurable quantity, say either angle θ or $\dot{\theta}$, angular velocity of system (A) for example, the response of such a control circuit will be *quadratic* as follows: from equation (71). This response will thus measure either $\mu\theta^2$ or $\mu\dot{\theta}^2$ as the case may be. Thus, this particular contour transforms a linear response (θ or $\dot{\theta}$) into a quadratic one.

(2) *Inverse Problem.*

In practical applications, an inverse transformation, e.g., the transformation of a quadratic response into a linear one, is of a greater interest.

Such a case presents itself, for example, in the problem of antirolling stabilization of ships by activated tanks. From the standpoint of dynamics of the controlled process (3) it is desired to control the rate of flow $\dot{w}$ of ballast so as to have it continuously in proportion to, and in phase with $\ddot{\theta}$—the angular acceleration of rolling motion.

The law of control imposed by Dynamics of the control process is thus: $\dot{w} = a\ddot{\theta}$.

As explained in section 17, a control of this kind is produced by using $\dot{\theta}$ as primary control and $\dot{w}$ as secondary, or "follow up" control.

The usual methods of measuring the rate of flow (e.g., Pitot tube) give, however, a quadratic response $\dot{w}^2$ whereas for this particular law of control a linear response ($\dot{w}$) is desired.

A non linear control circuit effecting a functional transformation (from a quadratic law to a linear law) is thus necessary. The generating function $f(x) = ax^{-\frac{1}{2}}$ fulfills this requirement. In fact,

$$k(x) = \int f(x)dx = a \int x^{-\frac{1}{2}} \cdot dx = 2ax^{\frac{1}{2}} + \text{Const.}$$

The direct response x is quadratic in terms of $\dot{w}$, that is $x = b\dot{w}^2$. However, upon the emergence from the "transducer"—the non linear control circuit with the above form of generating function—the response is:

$$k(x) = 2a(b\dot{w}^2)^{\frac{1}{2}} + \text{Const.} = A\dot{w} + \text{Const.} \qquad (72)$$

The transduced response $\dot{w}$ is thus linear—the result, which we were looking for.

(3) *Algebraic Generating Functions with Positive Exponents.*

By using different generating functions, either algebraic or transcendental, the method of the integrating contour offers an infinite variety of non linear controls, the properties of which can be readily analyzed in each particular case. For practical purposes already a simple algebraic function $y = x^n$ is capable of presenting the problem of non linear controls in a rather general manner.

Two examples of this function for $n = 1$ and $n = -\frac{1}{2}$ have already been considered.

Since the problems of control are always studied within a certain range, or interval, of variation of the corresponding

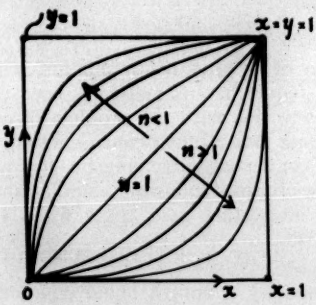

FIG. 4.

basic variable and, since this interval, by a suitable normalization, can be considered as contained between limits 0 and 1, one is thus led to investigate the behavior of function $y = x^n$ in that range in terms of its parameter n; this, in fact, determines a square since for $x = 1$, $y = 1$ whatever n may be. The general form of curves $y = x^n$ is shown in Fig. 4 for n varying between 0 and ∞.

For n increasing from $n = 1$ to ∞ the parabolic curves $y = x^2$, $y = x^3 \cdots$ grow slower and slower at the beginning of the interval (0, 1) and faster and faster at its end as $n \to \infty$.

For n decreasing from $n = 1$ to zero, on the contrary, these curves increase faster and faster in the beginning and slower and slower at the end of the interval as $n \to 0$. In the first case ($1 < x < \infty$), the control action will be negligible for small disturbing deviations and will grow up very quickly towards the end of the control range. In the second case ($0 < x < 1$), the control action will rise sharply for small deviations and will grow rather slowly for larger ones.

In this latter case, the features of a non linear control remind those of a discontinuous "on" or "off" (French: "tout ou rien") control—while still retaining the features of continuity (e.g., availability of Taylor's expansion, possibility of using higher time derivatives in the control, etc.). These particular non linear controls, bridge the gap, so to say, between the continuous and the discontinuous control systems.

(4) *Algebraic Generating Functions with Negative Exponents.*

An example of a generating function with a negative exponent, viz., $f(x) = ax^{-1}$ has already been considered (example 2) in connection with the problem of conversion of a quadratic response into a linear control action.

Problems of converting cubic, quartic, etc., responses into a linear one can be obtained following a similar line of argument. A particularly interesting case arises when the generating function is $y = \frac{1}{x}$ (i.e. $n = -1$) that is when the integrating contour is limited by a branch of an equilateral hyberbola referred to axes of coördinates as asymptotes. In such a case the response $k(x)$ *is logarithmic.* One could develop a series of interesting conclusions in connection with such control systems having a logarithmic response, particularly for more complicated problems in which the resultant response proportional to a *product* of a number of component control actions is aimed at. These few examples give probably a sufficient idea as to the problem of predetermination of response of a given non linear control system.

Note: The method of integrating contour is not the only one that can be applied. For instance one could easily apply for the same purpose an "integrated luminous flux" actuating a

photo electric cell for example. The double-shaded area of flux in Fig. 3 in such case will be the opening through which a luminous flux is admitted to a suitable optical condensing system transmitting the integrated value of luminous flux to a photo cell. The "integrating contour" in this case is replaced by a suitably bounded "integrating shutter" or an equivalent arrangement.

27. *Stability of Non Linear Control Systems. Liapounoff's Theorem.*

The question of stability of non linear systems is too complicated to be treated here (19). It has been investigated by Liapounoff (12). We propose to review briefly a particularly simple case when non linearity in system (A) is localized in the "sping constant" term (free of derivatives) which is now $A_0(\theta)$ instead of $A_0\theta$ considered for linear systems where $A_0(\theta)$ designates a certain function of θ. The non linear control is given by a function $S(\theta)$.

Linear operators cannot be used here and one has to investigate the differential equation directly. One has:

$$A_2\ddot{\theta} + A_1\dot{\theta} + A_0(\theta) + S(\theta) = 0. \tag{73}$$

The point θ_0 is a point of equilibrium if

$$A_0(\theta_0) + S(\theta_0) = 0. \tag{74}$$

In fact, if the system is brought to this point with zero velocity and acceleration, it remains at this point by virtue of equation (74). Graphically this point θ_0 corresponds to the intersection point of curves $y = A_0(\theta)$ and $y = -S(\theta)$.

The question of stability can be investigated by giving the variable θ a small increment x around the θ_0 value.

Since $\theta = \theta_0 + x$, $\dot{\theta} = \dot{x}$; $\ddot{\theta} = \ddot{x}$; which gives:

$$A_2\ddot{x} + A_1\dot{x} + A_0(\theta_0 + x) + S(\theta_0 + x) = 0. \tag{75}$$

Developing functions $A_0(\theta)$ and $S(\theta)$ in Taylor's series around θ_0 one has:

$$\left.\begin{aligned} A_0(\theta) &= A_0(\theta_0) + A_0{}^1x + A_0{}^2x^2 + \cdots \\ S(\theta) &= S(\theta_0) + S^1x + S^2x^2 + \cdots \end{aligned}\right\} \tag{76}$$

where:

$$A_0^1 = \left(\frac{dA_0}{dx}\right)_{\theta_0}; \qquad A_0^2 = \frac{1}{2!}\left(\frac{d^2A_0}{dx^2}\right)_{\theta_0}; \qquad \cdots$$

$$S^1 = \left(\frac{dS}{dx}\right)_{\theta_0}; \qquad S^2 = \frac{1}{2!}\left(\frac{d^2S}{dx^2}\right)_{\theta_0}; \qquad \cdots.$$

In view of equation (74), equation (75) becomes:

$$A_2\ddot{x} + A_1\dot{x} + (A_0^1 + S^1)x + (A_0^2 + S^2)x^2 + \cdots = 0. \quad (77)$$

If the departure x is small enough, curves A_0 and S are approximately linear in this range; in such a case A_0^2, A_0^3, S^2, S^3 are very small in comparison with A_0^1 and S^1 and the equation of the first approximation is:

$$A_2\ddot{x} + A_1\dot{x} + (A_0^1 + S^1)x = 0. \quad (78)$$

If $A_0^1 > 0$, the addition of S^1 does not introduce any new features. If, however, $A_0^1 < 0$, the stability can be maintained if $S^1 > |A_0^1|$, where $|A_0^1|$ is absolute value of A_0^1.

Thus, a control system $S(\theta)$ can provide an artificial stability to an essentially unstable system, *if the gradient of the control action at the point of equilibrium is greater than the absolute value of the negative gradient causing the instability.* The justification of this method is given by the Liapounoff's theorem (12, 19) which states: when real parts of the roots of the characteristic equation of the first approximation are different from zero, the latter gives always a correct answer to the question of stability of the original problem governed by a non linear differential equation. If not, the differential equations of the first approximation are unable to answer this question.

For non linear systems there may be several points of equilibrium which means that the curves C and S have several points of intersection. The above criterion can be used for the purpose of determining which of these points of equilibrium are stable and which are unstable.

When the criteria of the first order are not applicable, it is necessary to apply the criteria of higher orders and this complicates the problem very appreciably (12), (19).

Furthermore, if the range of stability is relatively small, it may happen that the system, while being stable for small disturbances, may leave the stable range under the effect of a relatively large disturbance and reach, thus, the unstable zone, through which it will pass until it reaches another stable zone; it may either acquire a position of a stable equilibrium in this zone or may start oscillating in the unstable zone between the two stable ones.

Phenomena of the so-called "hard" and "soft" self-excitation of electronic circuits give an example of such complicated conditions (19). Similar phenomena can be observed under special conditions in dynamical systems with non linear controls, if the non linear characteristics are strongly curved in the vicinity of the equilibrium point.

Similar conditions occur also when the non linearity occurs in higher time derivative terms. Thus, for instance, if the characteristics of A_1 and S_1 (terms associated with $\dot{\theta}$) are functions of the departure θ, conditions of damping of the controlled system $(A + S)$ may depend on this departure θ. In this manner rather complicated "relaxation oscillations" appear spontaneously in a controlled system of this kind.

Investigation of these complicated phenomena requires the application of methods of non linear mechanics (19) which is outside the scope of this paper.

28. *Effect of a Retarded Control in Non Linear Systems.*

The general procedure remains the same (Section 21) as in the case of linear control problems with that difference, however, that in view of the non linearity, the Taylor's expansions are more complicated in this case, as will be shown. Consider the same simple example when the control term of the non linear problem is of the form $S(\theta)$. In the corresponding linear problem it is $S = a\theta$. The retarded control action is then:

$$\bar{S}(\theta) = S(\theta) - \Delta t S'(\theta) + \frac{\Delta t^2}{2!} S''(\theta) - \cdots. \qquad (79a)$$

The peculiarity which presents itself in this case is in that $S'(\theta)$, $S''(\theta)$ $\cdots$ are the *time derivatives*, whereas $S(\theta)$ does not

depend on time directly but only through intermediary of θ which is a function of time.

The Taylor's expansion in this case is:

$$S(\theta(\bar{t})) = S(\theta(t - \Delta t))$$
$$= S(\theta(t)) - \Delta t \frac{dS}{dt} + \frac{\Delta t^2}{2!}\frac{d^2S}{dt^2} - \frac{\Delta t^3}{3!}\frac{d^3S}{dt^3} + \cdots,$$

where

$$\frac{dS}{dt} = \frac{dS}{d\theta}\cdot\dot{\theta}; \qquad \frac{d^2S}{dt^2} = \frac{d^2S}{d\theta^2}\cdot\dot{\theta}^2 + \frac{dS}{d\theta}\cdot\ddot{\theta};$$
$$\frac{d^3S}{dt^3} = \frac{d^3S}{d\theta^3}\dot{\theta}^3 + 3\frac{d^2S}{d\theta^2}\dot{\theta}\ddot{\theta} + \frac{dS}{d\theta}\dddot{\theta}; \qquad \cdots.$$

This gives:

$$S(\theta(\bar{t})) = S(\theta(t)) - \Delta t\left(\frac{dS}{d\theta}\cdot\dot{\theta}\right)$$
$$\frac{\Delta t^2}{2!}\left(\frac{d^2S}{d\theta^2}\dot{\theta}^2 + \frac{dS}{d\theta}\cdot\ddot{\theta}\right)$$
$$- \frac{\Delta t^3}{3!}\left(\frac{d^3S}{d\theta^3}\dot{\theta}^3 + 3\frac{d^2S}{d\theta^2}\dot{\theta}\ddot{\theta} + \frac{dS}{d\theta}\cdot\dddot{\theta}\right) + \cdots. \qquad (79)$$

For linear controls considered in Part III,

$$S = a\theta; \qquad \frac{d^2S}{d\theta^2} = 0; \qquad \frac{d^3S}{d\theta^3} = \frac{d^4S}{d\theta^4} = \cdots 0.$$

This gives on cancelling a, equation (54).

Equation (79) is quite general; for example, for a non linear control of the form: $S = \frac{1}{2}a\theta^2$, considered in example (1), Section 26,

$$\frac{dS}{d\theta} = a\theta; \qquad \frac{d^2S}{d\theta^2} = a; \qquad \frac{d^3S}{d\theta^3} = \frac{d^4S}{d\theta^4} = \cdots = 0.$$

From equation (79) one has:

$$S(\theta(\bar{t})) = S(\theta(t)) - \Delta t\cdot a\theta\dot{\theta} + \frac{\Delta t^2}{2!}a(\dot{\theta}^2 + \theta\ddot{\theta})$$
$$- \frac{\Delta t^3}{3!}(3\dot{\theta}\ddot{\theta} + \theta\dddot{\theta}) + \cdots. \qquad (79a)$$

If Δt is small, one can neglect terms with Δt^2, $\Delta t^3 \cdots$; in this case the difference between the retarded and non retarded control actions is:

$$\bar{S} - S = -\Delta t a \theta \dot{\theta}. \tag{80}$$

It is seen that the first order term depends now not only on $\dot{\theta}$ as in the case of linear controls, but also on θ, that is, *on the point of the range at which the retarded process is considered.*

This constitutes an essential difference between the linear and non linear controls, inasmuch as the retarded action in the latter depends not only on the value of time lag but also on the functional relation between S and θ.

These peculiarities of a retarded action in non linear controls can be, perhaps, better understood from the following geometrical considerations (Figs. 5 and 6). Figure 5 shows

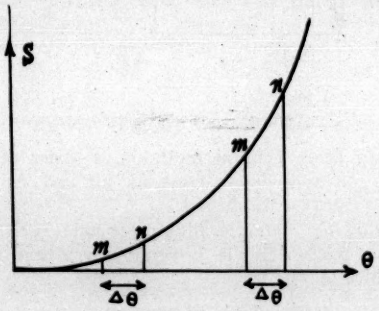

FIG. 5.

the control function of the type $S = a\theta^n$ for $1 < n < \infty$ and Fig. 6 when $0 < n < 1$ (Section 26, (3)). For the same lag $\Delta\theta$ (which is proportional to time lag Δt), the difference between the retarded and non retarded actions is measured by difference of ordinates n, m at the ends of the interval $\Delta\theta$ measuring the time lag Δt. In Fig. 5 this difference is small in the beginning of the range and large at its end, whereas in Fig. 6, on the contrary, this difference is large in the beginning of the range and small at the end.

It follows, therefore, that the effect of a given time lag in a non linear control system is *felt differently on the control action in different points of the range within which the control system operates*. These considerations are applicable to other non linear control terms $S_1(\dot{\theta})$, $S_2(\dot{\theta})$ etc., following a similar line of argument.

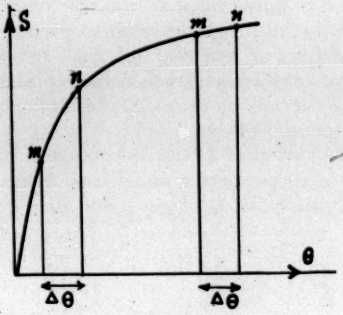

FIG. 6.

29. *Non Linear Systems Acted Upon by Non Linear Controls.*

Following these general methods, a series of interesting possibilities of non linear controls applied to non linear systems (A) can be established.

The following example illustrates one such possibility. Assume that system (A) be governed by a non linear differential equation of the form:

$$A_2\ddot{\theta} + A_1\dot{\theta} + A_1{}^1\dot{\theta}^2 + A_0\theta = 0. \tag{81}$$

The non linearity in this case is due to the term $A_1{}^1\dot{\theta}^2$. The prototype system in this case is clearly a pendulum which, in addition to the "velocity damping" $A_1\dot{\theta}$ also has a damping $A_1{}^1\dot{\theta}^2$ proportional to the square of velocity.

A control term $S_1{}^0 = a_1{}^0\dot{\theta}$ considered in example 1, Section 17, associated with a triangular integrating contour (Section 27) permits obtaining a control action—the control moment—of the form $m\dot{\theta}^2$.

The controlled process in this case becomes:

$$A_2\ddot{\theta} + A_1\dot{\theta} + (A_1{}^1 + m)\dot{\theta}^2 + A_0\theta = 0. \tag{82}$$

If the coefficient m of intensity of such a non linear control and its connection are so adjusted as to have $A_1^1 + m = 0$ the controlled process becomes:

$$A_2\ddot{\theta} + A_1\dot{\theta} + A_0\theta = 0,$$

that is, is given by a *linear* differential equation. It is thus possible to transform a non linear "natural" system (A) into a linear "controlled" one $(A + S)$ by cancelling out by means of a non linear control term $m\theta^2$ the non linear term $A_1^1\theta^2$ of the original system (A).

Conversely, for an opposite connection, the initial non-linearity can be further increased by such a control, or entirely new non linear terms can be thus artificially introduced.

30. *Applications of Non Linear Control Systems.*

From the preceding brief review of properties of non-linear control systems, it is apparent that they offer a far greater variety of different features than the linear controls. In fact, in linear control systems, the "transduced," or output, quantity stands in a simple linear, or proportional, relation to the quantity applied at the "input end" of the control system.

In a non linear control system effecting *a functional transformation*, the input quantity and the corresponding output quantity, transduced by such a system, can be related by any functional relation which one may select in advance.

In this manner, linear control systems stand with respect to non linear ones in the same relation in which a simple lever system transforming one rectilinear motion into other proportionally reduced (or amplified) rectilinear motion, stands with respect to a more complicated mechanical mechanism involving cams, etc., by which to one motion, e.g., the rectilinear one, at the input end, is related an entirely different curvilinear motion at the output end of the mechanism, or vice versa.

The above examples of transformation of one functional law into another law (e.g., linear response into a quadratic, or cubic or logarithmic, etc., response or vice versa) have been already mentioned. A still greater variety of applications of non linear controls lies in connection with the problem of

automatic computation in which a certain "output function" is obtained as the result of application of a *functional control operator* to the input function, or functions. As one application of this, the mechanical integration of the various differential equations can be cited (20), (18).

Still more numerous are the practical problems of automatic computation in which multiplication, division, powers, roots, logarithms, trigonometric etc., functions of different input quantities are to be combined according to a certain law imposed by the problem. The use of suitable non linear transducers considered in the above examples permits elimination of complicated and expensive mechanical computing mechanisms used at present.

An objection usually raised against the use of electronic circuits for this purpose, that they are insufficiently accurate or stable for such application, is hardly justified at present. As a matter of fact, an accuracy of about 1 per cent. can be easily obtained even now without any particular precautions. If there is a need for a still greater accuracy, it can be obtained by using special stabilized circuits, etc. In many practical applications of automatic computers, an accuracy of 1 per cent. is already sufficient.

An important advantage of such non linear circuits used in connection with problems of automatic computation is in that they are capable *not only of computing but of controlling as well*, as this was mentioned in connection with example (2), Section 26.

It is hoped that this brief review of non linear control problems will be of interest to both the inventors and designers engaged in special problems of control transcending the reach of the linear control methods now in use.

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The Measurement of Body Currents.—ROBERT S. SCHWAB (*Electrical Engineering*, Vol. 60, No. 10). In these days of supersensitive amplifiers and recording apparatus, it is very tempting to define life and death in terms of electrical activity. Whether or not this concept is accurate, we can on present knowledge liken living tissue to a B-battery and dead tissue to a burned out generator. The function of living tissue, however, is so closely allied with its electrical activity that knowledge of the latter has given us better understanding of the working of the human body. One type of body currents is the rather static, direct current potential difference. It is this ever-present E.M.F. found in all living matter, small (few millionths of a volt), but constant as to polarity and relatively unchanging in time or species, that brings up the easy analogy of life to the B-battery. These D.C. potentials are always associated with the first and most important character of life, namely; absorbing energy, growth, and change in structure. We call these activities metabolism and catabolism. The second type of body currents is that associated only with the animal kingdom—the electrical activity of locomotion. It is nature's second and perhaps most interesting attribute of life. Contractile tissue is nature's first intrinsic movement in an environment of physical forces all in motion, such as wind, tides, rivers, etc. This tissue, when left alone "lives," i.e. has its D.C. potentials. But, when it is stimulated from without or from within, an astonishing change takes place. It moves actively, withdraws, strikes out. Associated with this defensive or aggressive movement there is an alternating current wave, a diphasic spike, which is lacking in the motionless tissues. A third type of electrical activity is associated with the more highly developed types of contractile tissue. Cilia, tubular digestive tracts, and heart muscle have a "beat" of their own and depend no longer on outside stimuli for their continuous, regular contractions and relaxations. Every "beat" has its accompanying bursts of diphasic waves. The fourth type of body electrical currents is that associated with the complex tissue that makes up the central nervous system of animals. Here, as the function is continuous during life, the ganglia and brain cell tissues are ever-active electrically and show no periods of rest in the manner of muscle and nerve.

R. H. O.

CORRECT DESIGN OF RIGID STRUCTURES.

BY

OTTO GOTTSCHALK.[1]

SYNOPSIS

Statics as well as other branches of applied natural science need the visual experiment as a logical and true guidance toward efficient and correct methods of structural analysis, avoiding unnecessary complications and correcting erroneous interpretations.

As an example, the simple visual experiment indicated below for crossed beams brings out statical conditions that are not being taken into consideration generally, in spite of their importance for the economy and safety against loads and particularly shocks in floor and vault design. At the same time it gives the natural basis for designing two or four way reinforced slabs.

The simple equations (1) to (4) represent all the mental tools necessary for the correct analysis of structures; they give immediately, results which at present are obtained by the most advanced abstract methods and only by a long series of balancing moments or rotations.

Schemes guiding to proper applications of the visual experiment are given and the conveniences shown of working with fixed points and of analyzing portal structures as straight continuous beams.

DEFINITION OF CORRECT DESIGN.

In correct design or statical calculation of structures we have to keep apart clearly two fundamentally distinct parts: The structures, composed by members of definite resistance, and the loads with the stresses they produce. Evidently the loads are quantities entirely distinct and independent on the structures and it is precisely one of the fundamental errors most customary and of greatest consequence and source of unnecessary complications to treat the loads as if they were part of the structures. No method of design then is correct which tries to make loads or stresses a part of the analysis proper of structures.

In structural analysis it must be kept in mind that structures are geometrical bodies, which can be analyzed rationally only by geometrical methods. Knowing, as is

[1] Buenos Aires, Argentine Republic.

usual in statical calculations of elastic structures, the length L, the moment of inertia I and therefore the specific rigidity $k = I/L$ of the members the correct way of analysis of a structure is to mark in the plan of it at both ends A and B of its members AB the ratio of end rotation m and m' for rotations originating at A and B respectively (equation 1) and the corresponding relative stiffness S and S' (equation 2) and to produce or imagine a unit geometrical displacement at the section the stress of which is wanted. This way we obtain the rotations at the rigid joints of the structure $\theta_A = f'/L$ at A and $\theta_B = f/L$ at B, which mark the tangents in A and B of the influence line between A and B and permit us to calculate the ordinate y at any point of AB (equation 3) and the tensions Py produced by a load P at that point.

For its fundamental importance may we repeat then as a sane criterium of correct methods of structural design:

1. *The correct statical calculation of buildings is being done in two strictly separate steps: the analysis of the structure and the computation of stresses produced by loads.*

2. *The structural analysis is a strictly geometrical operation, based upon the equations (1) to (4) given below, and applied in accordance with the conditions of equilibrium and the geometrical relations in deformed models,* as illustrated in this paper (Figs. 3, 5 and 7).

THE GEOMETRICAL ANALYSIS OF STRUCTURES.

The correct geometrical analysis of structures has been presented by the writer before [2] in his paper 2000; the chief results obtained in that paper may be resumed and completed shortly, supposing again that the structure to be analyzed consist of members, each of constant section; extending, however, the scope of that paper,[2] members may be horizontal, vertical or inclined under any angle. It is further to be remembered that all deformations are infinitely small, although magnified for purposes of illustration in the figures

[2] "Structural Analysis Based upon Principles Pertaining to Unloaded Models" by Otto Gottschalk, Paper 2000, *Transactions Am. Soc. C. E.*, 103 (1938), p. 1019.

below, so that (Fig. 1) the length of a curve AB after deformation is equal to the original straight line AB.

Let (Fig. 1):

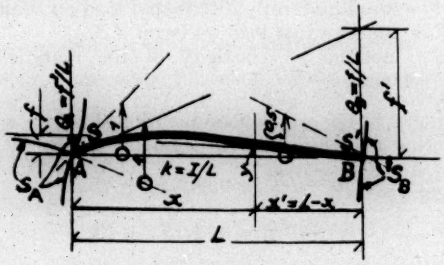

FIG. 1. Rotation "1" at end A.

$k = I/L$, the specific stiffness of a member AB.

f and f' the intercepts at A and B of tangents in B and A.

$\theta_A = f'/L$ and $\theta_B = f/L$ rotations of tangents at A and B.

$m = f/f'$ and $m' = f'/f$ ratios of end rotations, at the curve AB, which may vary between $+1$ and -1.

S and S' the resistance to rotation at the left end A and right end B of the member AB.

S_A and S_B the sum of the relative stiffness values of all the members meeting at the left and right end of AB, but not including S and S' of the member AB itself.

$K_A = S + S_A$ and $K_B = S' + S_B$.

x = abscissa of the influence line, measured from the left support; $x' = L - x$.

y = ordinate at any point x of the influence line, h = ordinate in center of AB and y_m the mean value of any infl. line.

i and i' ordinates at $x = L/3$ and $x' = 2L/3$ respectively.

e and $e' = L - e$ abscissae of inflection points of influence curves.

P_d = force required to displace the ends A and B with respect to each other, generally in perpendicular direction to AB.
T = thrust at feet of frames or arches.
W = total uniformly distributed load on beam AB.
P = concentrated load on beam AB.
E = modulus of elasticity of the material of the structure.

As shown in the paper 2000 mentioned before we obtain the following 4 fundamental equations:

$$m = f/f' = 0.5k/(S_B + k)$$

and

$$m' = f'/f = 0.5k/(S_A + k), \tag{1}$$

$$S/k = 1 - 0.5m \quad \text{and} \quad S'/k = 1 - 0.5m', \tag{2}$$

$$y = xx'(xf + x'f')/L^3, \tag{3}$$

$$P_d = 6Ek(f' - f)/L^2. \tag{4}$$

The four simple equations (1) to (4) may be easily enough memorized and are all the armor required for the natural geometrical analysis of structures, taking the place of the enormous burden of methods and shortcuts accumulated in classical statics. It must be remembered that the most useful of those shortcuts are based upon the geometrical properties of deformation curves, which thus are being automatically taken care of in the geometrical analysis.

As an example we may cite the moment distribution introduced by Prof. Hardy Cross [3] a method of exceptional merits and probably one of the most important advances obtained within the abstract conception of statics. It will easily be seen that the "carry over factor" used in that method is identical with the ratio m of endrotations (equation 1), developed by the author [4] from the experiments and that the simple S values (equation 2) make unnecessary the series of distribution.

[3] "Continuous Frames of Reinforced Concrete" by Hardy Cross and Newlin Dolbey Morgan, New York, 1932.

[4] "Mechanical Methods of Stress Analysis," *Eng. News Record*, Jan. 22, 1931, p. 162.

Another example is the method of interchanged third perpendiculars widely applied especially by former authors to fixed points in continuous structures, which will be referred to in the chapter on fixed points.

SPECIAL CASES OF THE FUNDAMENTAL EQUATIONS.

For special cases the following values are convenient to remember.

The ordinate h of the influence curve (Fig. 1) in center of AB and y_m, medium ordinate for curve AB are

$$h = (f + f')/8 = 9(i + i')/16, \tag{5}$$

$$y_m = 2h/3 = (f + f')/12 = 3(i + i')/8, \tag{6}$$

$$i = 2(f + 2f')/27, \qquad i' = 2(2f + f')/27, \tag{7}$$

$$f = 9(i' - i/2), \qquad f' = 9(i - i'/2). \tag{8}$$

The abscissa e of the inflection point is

$$e = L(2 - m)/3(1 - m) \tag{9}$$

and may be determined graphically as the intersection with the axis AB of a straight line joining the $L/3$ points on the end tangents.

Be it also remembered as a convenient aid in drafting influence lines that the tangent to center of the influence curve AB is a straight line crossing at $f'/4$ at A and at $f/4$ at B. For other graphical methods the paper 2000 [2] is referred to, for special cases of equations (1) to (9) the Table I will prove convenient.

TABLE I.

Special Cases of General Influence Lines for Correct Analysis.

Beam AB.	m.	m'.	S/k.	S'/k.	yL^2.	h.	y_m.	e/L.
Free at A.....	—	0.5	—	0.75	$xx'f(L + x)/2$	$3f/16$	$f/8$	0
Free at B.....	0.5	—	0.75	—	$xx'f'(L + x')/2$	$3f'/16$	$f'/8$	1
Fixed at A....	—	0	—	1	$x^2x'f$	$f/8$	$f/12$	1/3
Fixed at B....	0	—	1	—	xx'^2f'	$f'/8$	$f'/12$	2/3
$f = f'$.....	1	1	0.5	0.5	$xx'fL$	$f/4$	$f/6$	—
$f = -f'$.....	−1	−1	1.5	1.5	$xx'f(x - x')$	0	0	0.5

STRESS EQUIVALENTS OF DEFORMATIONS.

As has been stated at the beginning, abstract stresses do not form part of the correct design of structures and the equivalent bending moment equations 10 to 15 are being cited merely for the convenience in comparative studies. Let M_A and M_B the bending moments at A and B respectively corresponding to rotations θ_A and θ_B and $n = M_A/M_B$ and $n' = 1/n = M_B/M_A$:

$$\theta_B = (M_A + 2M_B)/6Ek; \qquad \theta_A = (2M_A + M_B)/6Ek, \quad (10)$$

$$f = (M_A + 2M_B)/6EkL; \qquad f' = (2M_A + M_B)/6EkL, \quad (11)$$

$$\left.\begin{aligned} m &= (M_A + 2M_B)/(2M_A + M_B) = (n + 2)/(2n + 1) \\ m' &= (2M_A + M_B)/(M_A + 2M_B) = (2n + 1)/(n + 2) \end{aligned}\right\}. \quad (12)$$

$$M_A = 2Ek(2f' - f)/L; \qquad M_B = 2Ek(2f - f')/L, \quad (13)$$

$$S/k = 3n/8(2n + 1); \qquad S'/k = 3/8(2 + n), \quad (14)$$

$$e/L = n/(n - 1); \qquad e'/L = 1/(1 - n) = n'/(n' - 1). \quad (15)$$

THE DISPLACEMENTS "1."

In Fig. 2 have been shown the geometrical definitions of the principal unit displacements. The heavy lines refer to the simple beam AB, the dotted curves to a beam AB restrained at ends. Lifting (or lowering) A to A' by "1" without moving any other support or joint of the structure (Fig. 2*a*) we obtain the influence line of the reaction R_A; lifting (or lowering) by "1" the structure to the left of A but leaving B and all parts to the right of B at the original level (Fig. 2*b*) produces the influence line for shear S_E at any section E between A and B. Finally, rotating by $a/a = 1$ (Fig. 2*c*) at E the parts EA and EB without moving A and B with respect to each other, produces the influence line of the bending moments M_E. The branches of influence line outside of AB in Figs. 2*b* and 2*c* and outside of the two spans joining at A in Fig. 2*a* have the general form shown in Fig. 1.

ORDINATES OF COMBINED INFLUENCE LINES.

The equations of ordinates of combined influence lines have been summed up in the Table II; for the simple beam

is at R_A (Fig. 2*a*) $f = 1$, $f' = 0$, at S_E (Fig. 2*b*) $f = 0$ and at M_E (Fig. 2*c*) $f = a$ and $f' = a'$.

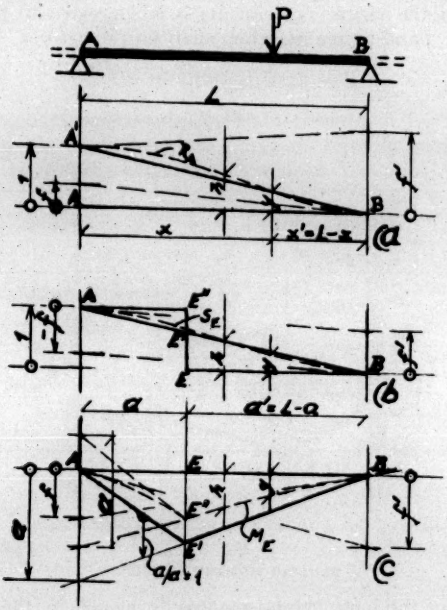

FIG. 2. Unit displacement for (*a*) reaction R_A, (*b*) shear S_E and (*c*) bending moment M_E.

MECHANICAL ANALYSIS FOR CORRECT DESIGN.

In former publications [5] the author has explained the visible model analysis and the instrument [6] "Continostat" devised for working accurately with large deformations, clearly

[5] "Mechanical Calculation of Elastic Systems," July 1926, JOURNAL OF THE FRANKLIN INSTITUTE, "The Experiment in Statics," February 1929, ibid.

[6] "Mechanical Structural Analysis by the Moment Indicator," A. C. Ruge and E. O. Schmidt, Paper 2050, *Transactions*, 104 (1939), p. 1681, discussion by Otto Gottschalk.

visible and readable, so it will be sufficient to reproduce a few examples which are specially appropriate to guide the correct application of the geometrical analysis as the outcome of the visible experiment; it is suggested to have the Figs. 3, 5 and 7 present, when analyzing structures.

TABLE II.

Ordinates of Combined Influence Lines.

Type of influence curve.	Fig.	General ordinate y.	h in center.	y_m medium.
General	1	$xx'(xf + x'f')/L^3$	$(f + f')/8$	$(f + f')/12$
Reactions	2a	R_A: $x'[1 + x(xf + x'f')/L^3]/L$ R_B: $x[1 + x'(xf + x'f')/L^3]/L$	$(3 + f + f')/8$	$(5 + f + f')/12$ $= \frac{2}{3}h + \frac{1}{6}$
Shear S_E	2b	$x < a$: $-x/L + N$ $x > a$: $x'/L + N$ $\therefore N = xx'(fx + f'x')/L^3$	$a < a'$: $(3 + f + f')/8$ $a > a'$: $(-3 + f + f')/8$	$\frac{a' - a}{2L} + \frac{f + f'}{12}$ $= \frac{1}{6} + \frac{2}{3}h - \frac{a}{L}$
General bending moment M_E	2c	$x < a$: $a'x/L - N$ $x > a$: $ax'/L - N$ $x = a$: $aa'(2aa' + af + a'f')/L^3$ $\therefore N = xx'[(a-f)x + (a'-f')x']/L^3$	$a < a'$: $a/2 - N$ $a > a'$: $a'/2 - N$ $\therefore N = (f + f' - L)/8$	$a < a'$: $\frac{2}{3}h - \frac{a}{6L}(3a - L)$ $a > a'$: $\frac{2}{3}h - \frac{a'}{6L}(3a' - L)$
Bend. mom. in center $a = a' = 0.5L$	2c	$x < 0.5L$: $0.5x - N$ $x > 0.5L$: $0.5x' - N$ $\therefore N = \frac{xx'}{L^3}\left(\frac{L^3}{2} - fx - f'x'\right)$	$(L + f + f')/8$	$(L + 2 + 2f')/24$ $= \frac{2}{3}h - \frac{L}{24}$

BENDING MOMENT ANALYSIS.

One of the most helpful devices developed by the author for the visible mechanical analysis with real or imaginary models is the moment clamp, which joins two splines rigidly under the angle corresponding to unit rotation which varies but little [5] over a relatively large part of a span AB. In the elastic center of the span it is 126° 52′. When drawing correct influence lines of bending moments one may imagine the beam broken under that angle, with the apex of the angle placed at the section at which the bending moment is wanted (Fig. 3).

The third point ordinates i and i' of the influence curve of the bending moment M_C (Fig. 3a) thus obtained, expressed

by the end rotations θ_A and θ_B and vice versa, are as follows:

$$i = L[3 + 4(2\theta_A + \theta_B)]/54;$$
$$i' = L[3 + 4(\theta_A + 2\theta_B)]/54, \qquad (16)$$

$$f/L = \theta_B = 9(2i - i')/2L - 1/4;$$
$$f'/L = \theta_A = 9(2i - i')/2L - 1/4. \qquad (17)$$

The equations (16) and (17) are convenient for figuring end tangents of the broken influence lines in mechanical analysis be it with visible or microscopical deformations.

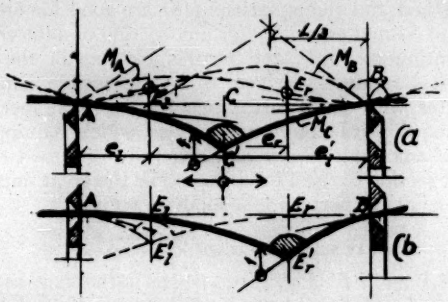

FIG. 3. Mechanical definition of fixed points as (*a*) inflection points of M_A and M_B influence curves. (*b*) Places E_l and E_r of E for which tangent horizontal at B and A respectively.

FIXED POINTS.

When the rotation clamp (Fig. 3*a*) stays in the central part of AB the tangents at A and B point below the axis AB; when it approaches one support the tangent at the other support turns upward until at certain positions at E_l near A and E_r near B it becomes horizontal, indicating that vertical loads to the right of B or to the left of A respectively do not influence any the bending moment at E_l and E_r, and that E_1 and E_r are the left and right fixed points of the span AB.

As shown in Fig. 3*a* E_l and E_r are identical with the inflection points of the M_A and M_B influence curves. We may then extend the equation (9) of the abscissæ e_l and e_r

of the fixed points E_l and E_r as follows:

$$\left.\begin{aligned} e_l &= \frac{1-2m'}{1-m'} \times \frac{L}{3} = \frac{mr_A}{1-r_B+mr_A} \\ e_r &= \frac{2-m}{1-m} \times \frac{L}{3} = \frac{1-r_A}{1-r_A+m'r_B} \end{aligned}\right\}, \qquad (18)$$

where $r_A = S_A/K_A$ and $r_B = S_B/K_B$.

In Fig. 3*a* is E_l at the intersection with the axis AB of the straight line passing through $L/3$ points of end tangents from rotation at B and similarly for e_r from rotation at A. This method and the equations (18) are good for any value of S_A and S_B and especially for any section of adjacent spans of a continuous beam and in this case again the correct geometrical analysis derived from the visible experiment substitutes with great advantage the classical methods of interchanged third points and others, which require a considerable amount of work and generally are good only for constant section over all its spans. For the great importance of fixed points in portal frame analysis see Fig. 7.

TWO WAY BEAMS AND SLABS. Correct Influence Lines.

Let AA' and BB' (Fig. 4) two beams in the same horizontal plane, crossing each other under a right or nearly right angle and rigidly joined at E; the lengths of the beams are L_a and L_b. Wanted are the influence lines of bending moments M_a and M_b at E.

In Fig. 5*a* are shown the two branches AA' and BB' of influence line of M_a, produced with the Continostat by the spline representing AA' broken by "1" at E; AA' is kept at A and A' from moving "upwards" and may by itself form the broken curve shown as dotted line. At a distance d "below" AA' in Fig. 5*a* is shown the beam BB', reproduced by an originally straight spline and supported at B and B' against "downward" movement. A spline of length d is introduced acting as a strut at E_a bending AA' "up" and BB' "down," so that $AE_a'A'$ and $BE_a'B'$ are the two branches of the correct influence line of M_a.

Inversely, for analyzing mechanically the influence line M_b we reproduce the beam AA' by an originally straight spline

as shown in the upper part of Fig. 5*b*, kept from moving "down" at A and A', and at a distance "d" below the beam BB' by splines broken by "1" at E_b, which by itself may originally form the dotted line shown below BB' and kept from moving "up" at B and B'. By a hook of length d at E_b the splines for AA' and BB' are tied to proper distance d at E_b, thus producing the two branches $AE_b'A'$ and $BE_b'B'$ of the influence line of M_b.

The Figs. 5*a* and 5*b* reveal facts not considered yet generally. In Fig. 5*a* for example the medium ordinate y_{ma}

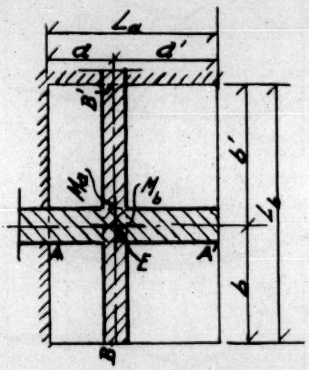

FIG. 4. Two way beams or slab.

of the branch AA' of the influence line M_a is considerably smaller than the y_{mb} of the branch BB', which means that in a cross $ABA'B'$ formed by two beams AA' and BB' rigidly joined at the intersection E the tendency is that the greater part of the load of one beam is chiefly supported by the other beam. Suppose in Fig. 5*a* were $L_a = L_b = L$, $a = a' = L/2$, $b = b' = L/2$, $k_a = k_b$ and the beams AA' and BB' loaded uniformly by W_a and W_b respectively. For free supports we have at the intersection $y_a = \frac{1}{2} \times L/8 = L/16$ and, the bending area of a free beam bent by a concentrated force by h in center being $5hL/8$, the bending moment in center of AA' is $M_a = (3W_a + 5W_b)L/64$, in other words

only 3/8 of the load on AA' itself and 5/8 of the load on the beam BB' is supported by beam AA'. The difference is even more marked when beams are restrained at ends. Suppose both beams were fixed at ends we would have $y_a = \frac{1}{2} \times L/16 = L/32$ and $M_a = (W_a + 3W_b)L/96$ so that on M_a of the beam AA' act only 1/4 of its own load and 3/4 of the load of

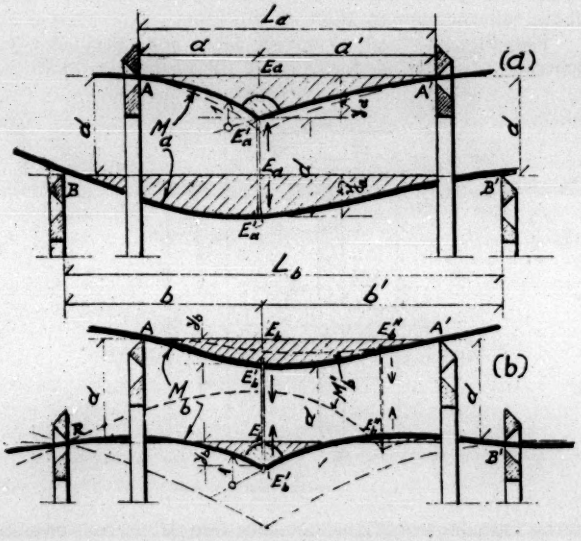

FIG. 5. Mechanical design of correct influence lines of interdependent (two dimensional) members at Fig. 4.

beam BB'. Similarly is the distribution of concentrated loads outside of the intersection itself, as can be seen from Fig. 5.

CORRECT DESIGN OF BEAMS AND GIRDERS.

In Fig. 6 has been shown a girder AB supporting 3 beams, Nos. 2, 3 and 4, all freely supported and rigidly joined with the girder AB as for instance in reinforced concrete. According to current practice and theory the influence line of

M_C in center C of AA' would result as shown in dotted curves, however due to the resistance of the beams themselves the branches 1, 2, 3 and 4 of the influence line M_C will be as shown in solid curves; the difference between dotted and solid curves is absorbed and transmitted to the supports by the beams Nos. 2, 3 and 4 directly. That difference may easily amount to 40 per cent. depending on the k/L^2 of the beams and girder, counting L_a and L_b (Fig. 6) from wall to wall.

When the outer ends of the beams 2, 3 and 4 are supported by flexible girders similar to AB instead of by columns or walls and when all the floor is being loaded at the same time we still have the possibility by proper arrangement of beam

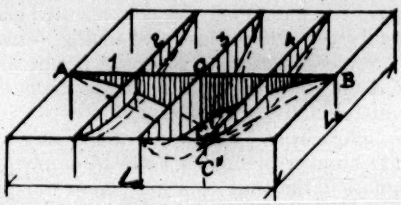

FIG. 6. Influence areas of M_c in center of simply supported girder AB crossed by beams 2, 3 and 4.

reinforcing with no or very little additional steel to resist local excess loads of up to those 40 per cent. at any one of those girders without harm to the coefficient of safety. Correct design should consider that possibility from the start and can do so easily, guided by Fig. 5.

In Fig. 5*b* the dotted curves on top of the heavy lines show the branches AA' and BB' of the influence line of M_b' at E_b'' near support A'. In this case the deflection of AA' becomes small when hooked to the clamp at the "lower" E_b'', and the BB' branch of the M_b' curve of influence becomes markedly negative like a support of a continuous or fixed beam. This reminds us that in two way beams or slabs great stiffness in one direction engenders negative bending the other way which may require negative reinforcement.

TWO WAY SLABS.

The principles explained in Figs. 4, 5 and 6 may be extended to grills and slabs, obtaining for the latter influence areas enclosing with the original plane the influence body much the same as the influence curve encloses with the original axis an influence area. Keeping present Fig. 5 and especially the M_b' curve Fig. 5*b* it will be possible to provide proper reinforcement against the secondary actions which may be neutralized for statical loads but which need be considered for structures exposed to heavy impact or shocks as in military structures. This theory of considering the two way slab as composed of resisting crossed beams is a sound basis for slab calculation and more satisfactory than current assumptions, especially for concentrated loads.

In flat slabs the action illustrated in Fig. 6 may become very considerable because the supposed beams are as deep as the girders and their distributing power under excess load will be therefore relatively important.

Valuable information [5] for correct design may be obtained in an infinite number of further cases by applying actually or in imagination the unit rotation clamp [6] in geometrical analysis the same as in Figs. 3 and 5, eliminating the present round about ways.

CORRECT DESIGN OF PORTAL FRAMES AND SKELETONS.

Figure 7 shows typical deformations of the model of the frame shown in Fig. 7*a*, fixed at *A* and *D*, when subjected to unit displacements, in order to serve as the natural basis and guide for structural analysis of frames and skeletons. The model is supposed to be composed of elastic splines joined at *B* and *C* by rigid clamps; for operating convenience it is handled upside down as shown in Fig. 7*b*, *c* and *d*. The column feet *A* and *D* are held at proper distance and under specified fixing angle by clamps mounted on a round rod, which at its ends is carried by roller bearings, so that it may freely move horizontally. It is recommended to recall those mechanical details and the position and pressures of the splines in Fig. 7 when deformed, as a means to convey the correct "feel" of frame action.

The influence line of sidesway (Fig. 7*b*) is produced by pushing the rod with the column feet A and D to the right by "1," so that the column feet stop at A' and D', being $AA' = DD' = 1$. The force necessary for that operation is $P = P_A + P_D$. The relation P_A/P may be figured in two ways as indicated in Fig. 7*b*, first according to the fundamental equation (4) which gives from the f and f' the trans-

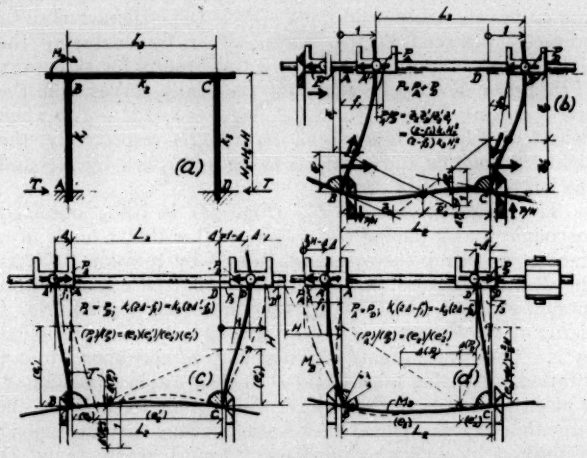

FIG. 7. Forces and geometrical relations in deformed models facilitate analyzing correctly structures as straight beams (Figs. 11 and 12). Letters (ε) refer to firm structures, $\bar{\varepsilon}$ to correction for side sway. In Fig. 7*d* deformations are reduced to 1/2.

versal thrust necessary for displacing A and D with respect to B and C and second, geometrically, using the fixed points; the required equilibrium of the vertical components at the diagonals between the fixed points of columns and beam give the singularly simple relation $P_A/P_D = \bar{\varepsilon}_2\bar{\varepsilon}_3'/\bar{\varepsilon}_2'\bar{\varepsilon}_1'$. The graphical solution is most simple, obtaining P_A and P_D as parts determined on the vertical unit "1" between the diagonals by the axis BC as shown in Fig. 7*b*. This is a most useful geometrical relation for correct analysis. In the

example No. 5 (Fig. 11) the resulting values from side sway have been analyzed by which the results obtained for the immovable structure have to be corrected.

The influence line of the horizontal thrust at the column feet is obtained by displacing by "1" one way (Fig. 7*c*) the column foot D to D', producing the dotted curves after which the rod will sway back by $\Delta = AA'$ until the horizontal thrusts at both column feet become equal, $P_A = P_D$, forming the curves shown in solid lines. Thus the structure may be correctly designed in two ways, either by analyzing the dotted curves (Fig. 7*c*) and correcting results for side sway (Fig. 7*b*) or directly by analyzing the heavy curves from the start, making $P_A = P_D$ or $k_1(2\Delta - f_1) = k_3(2\Delta' - f_3)$: when height of columns is unequal, H_1 and H_3 respectively, the right side of this equation has to be multiplied by H_1^2 and the left side by H_3^2.

The influence line of M_B (Fig. 7*d*) is being obtained mechanically by joining the splines at B with the angle of a right angle clamp increased or reduced by rotation "1"; as the deformations obtained this way would be excessive and inexact generally only rotations 0.5 are being applied (No. 4 clamp of the Continostat) and resulting ordinates multiplied by 2. The deformations shown in Fig. 7*d* correspond to 0.5 rotation. Keeping first A and D in their original position A is pushed towards the left by the diminishing force P_A at the same time D is being held back from following that movement of the rod by an increasing force P_D until, when A and D arrive at A' and D', the equilibrium $P_A = P_D$ has been obtained. Again we may correctly analyze M_B either imagining A and D retained in their original position, as indicated in dotted curves (Fig. 7*d*) and correcting for side sway (Fig. 7*b*) or directly with the sway corresponding to $P_A = P_D$, where in straight line analysis the Δ at A and D are on opposite sides of the line AD.

OBLIQUE STRUCTURES.

Figures 8, 9 and 10 illustrate as a typical example of structures with sloping members the frame ABC, hinged at A and C in Figs. 8 and 10 and fixed at C in Fig. 9, to be designed for horizontal and vertical loads. The structure is

conveniently analyzed as straight the sum of beam problems, which are the continuous beam $AB'C$ for vertical ordinates and two coinciding vertical beams $B''B$ and BB'' rigidly joined on top B and hinged on bottom B'', shown as one only axis BB''. The bending moment at B (Figs. 8 and 9) is obtained geometrically by the unit rotation $\theta_1' + \theta_2 = 1$, where

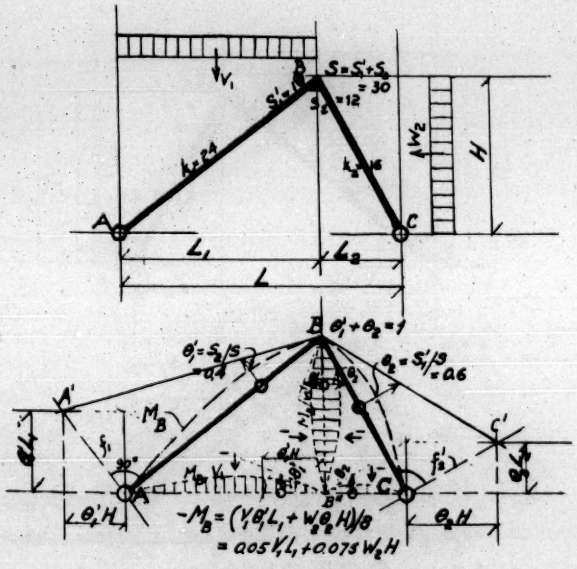

FIG. 8. Analysis for M_B of oblique frame.

$\theta_1' = S_2/S$ and $\theta_2 = S_1'/S$, being $S = S_1' + S_2$. The branches of influence lines to the left of BB'' are for loads on the sloping beam AB and to the right of BB'' for loads on BC.

In Fig. 10 have been shown similarly separated the branches for vertical and horizontal loading of the influence line for horizontal thrust T_A and T_C at A and C respectively; the axis for vertical loading is $AB''B$ and for horizontal loads $A'A''$. For vertical loading on either AB or BC is

$T_A = T_C$, for any horizontal load T is $T_C = T - T_A$. The ordinates of the influence line of axis $A'A''$ therefore are serving for both T_A and T_C; the ordinates for T_A are y and for T_C they are $1 - y$.

As shown in Fig. 10 the components of gyration at B around C and D give for a vertical load P on top B: $T_A = T_C$

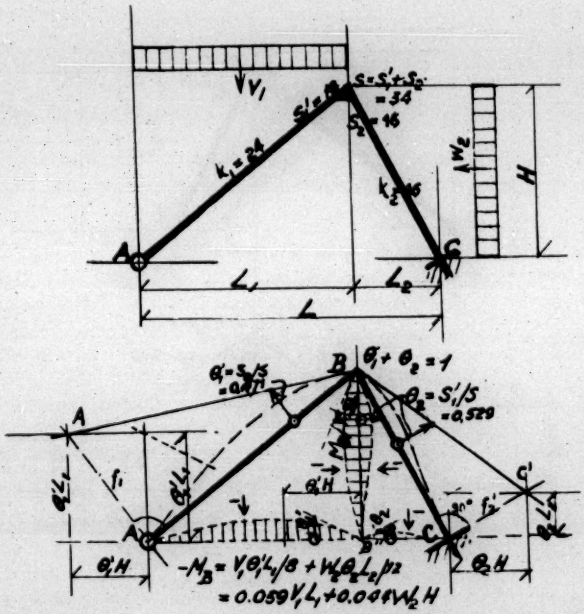

FIG. 9. Analysis for M_B of oblique frame fixed at C.

$= PL_1L_2/HL$ and for a horizontal load P at B is $T_A = PL_1/L$ and $T_C = PL_2/L$. For a horizontal load P at A is $T_A = P$ and $T_C = 0$ and for P horizontal at C is $T_A = 0$ and $T_C = P$. The $AB''C$ branch of the influence line for both T_A and T_C is the sum of a triangle of height L_1L_2/LH at B'' plus the curves produced by rotations θ_1' and θ_2 to the left and the

right of B'' and hinged at A and C; consequently for the branch AB'' we have $y_{m1} = L_1(L_2/L + S_2/4S)/2H$ and for the part $B''C$ is $y_{m2} = L_2(L_1/L + S_1/4S)/2H$. The influence lines for T_A at the vertical axis $A'A''$ (Fig. 10) are composed for AB, as shown to the left of $A'A''$, of a trapezoid of base

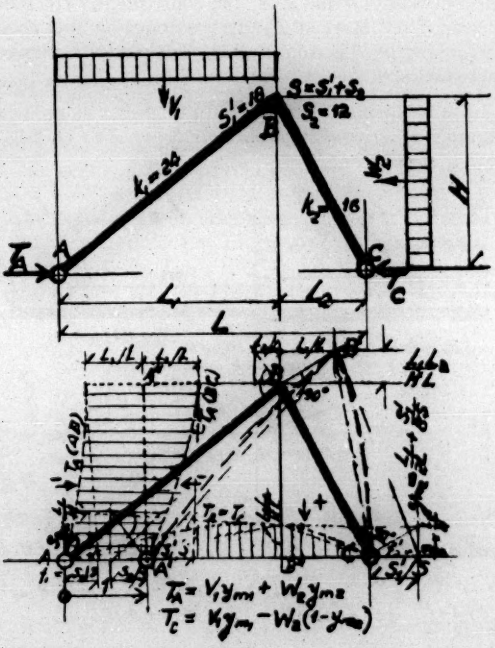

FIG. 10. Analysis for T_A and T_C of oblique frame.

"1" and top L_1/L minus a curve produced by a rotation $\theta_1 = S_2/SH$ and to the right of $A'A''$ of a triangle of base 0 and top L_1/L plus the curve produced by a rotation $\theta_2 = S_1'/SH$. We have, then, for horizontal uniform load on AB:

$$y_{m1} = 0.5(1 + L_1/L) - S_2/8S$$

and on BC:

$$y_{m2} = L_1/2L + S_1'/8S.$$

Again the geometrical method of analysis is essential for correct design by its simpleness and the intuition it affords; as an example in Fig. 10 the curve $A'B'$ carves out part of the area between AB and $A'B'$ and consequently the influence curve over $A'A''$ marked $T_A(AB)$ is concave; this check by merely looking at it is safer and easier than with numbers in abstract statics.

NUMERICAL EXAMPLES.

Example 1: Irregular continuous beam ABCD, Fig. 11*a*, fixed at A, cantilever CD.

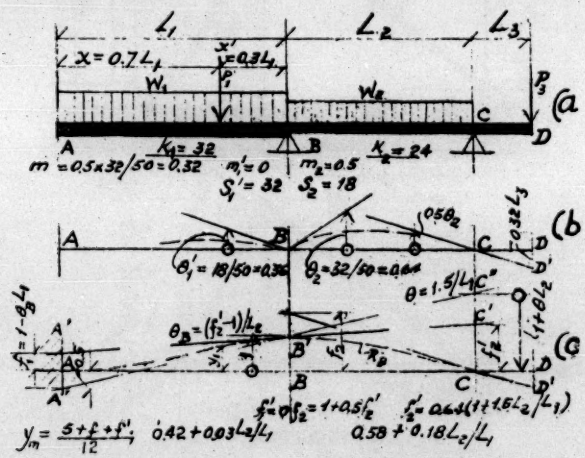

FIG. 11. (*a*) Continuous beam $ABCD$ fixed at A. (*b*) Analyzing M_B by rotation $\theta_1' + \theta_2 = 1$ at B. (*c*) Analyzing R_B by lifting $BB' = 1$.

Wanted: Bending moment M_B at support B.

Analysis: In Fig. 11*a* for beam AB, due to fixing at A, is to the left of B: $m_1' = 0$ (equation 2), $S_1' = k_1 = 32$ and to the right of B: $m_2 = 0.5$, $S_2 = k_2(1 - 0.5 \times 0.5) = 24 \times 0.75 = 18$; $S_1' + S_2 = 32 + 18 = 50$.

Rotation "1" at B (Fig. 11*b*) revolves BA by $\theta_1' = 18/50 = 0.36$, BC by $\theta_2 = 32/50 = 0.64$, CD by $\theta_C = 0.5\theta_2 = 0.32$, therefore acc. Table I: $y_{m1} = 0.36L_1/12 = 0.03L_1$; $y_1 = 0.7^2 \times 0.3 \times 0.36L_1 = 0.05292L_1$; $y_{m2} = 0.64L_2/8 = 0.08L_2$; $y_3 = -0.32L_3$.

Stress computation: $-M_B = 0.03L_1W_1 + 0.05292L_1P_1 + 0.08L_2W_2 - 0.32L_3P_3$.

Example 2: Irregular continuous beam, similar example 1.

Wanted: Reaction R_B at B.

Analysis: m and S values the same as in example 1.

Transversal displacement $BB' = 1$ (Fig. 11*c*) would bend the spline AB, if not retained at C, so that the tangent $A''B'C''$ marks $AA'' = -0.5$ at A and $CC'' = 1 + 1.5L_2/L_1$ at C. Bending the end at C'' down again to C will turn the tangent at B so that $CC' = f_2' = (1 + 1.5L_2/L_1)S_1'/(S_1' + S_2) = 0.64(1 + 1.5L_2/L_1)$; $f_1 = 1 - (f_2' - 1)L_1/L_2 = 0.04 + 0.36L_1/L_2$; $\theta_C = (1 + f_2'/2)/L_2 = 1.32/L_2 + 0.48/L_1$, $f_2 = 1.32 + 0.48L_2/L_1$.

According to Table II: $y_{m1} = (5 + f_1 + f_1')/12 = 0.42 + 0.03L_1/L_2$, $y_1 = (1 + (x'f_1 + x)x'/L_1^2)x'/L_1 = 0.7059 + 0.05292L_1/L_2$, $y_{m2} = 0.4967 + 0.12L_2/L_1$: $y_3 = -\theta_C L_3 = -L_3(1.32/L_2 + 0.48/L_1)$.

Stress computation: $R_B = W_1y_{m1} + P_1y_1 + W_2y_{m2} + P_3y_3$, with y values as calculated.

Example 3: "Endless" continuous beam, a continuous beam over an "infinite" number of spans (Fig. 12*a*).

Wanted: The exact bending moment M_E in center E of AB.

Analysis: In order to find the exact m and S at A and B it is sufficient to go back two spans to the left and right respectively, assuming outside of second support $S = 7k/8$ or at left (Fig. 12*a*) $S' = 7 \times 12/8 = 10.5$ and $S = 7 \times 24/8 = 21.0$ respectively, calculating with equations (1) and (2) the m' and S' going from left to right and the m and S starting at the right end and going left. Then at A we obtain [2] mathematically correct $m = 0.211$, $S = 17.892$ and at B: $m' = 0.367$, $S' = 16.332$.

If ends at A and B were free the break "1" in center E of AB (Fig. 12*b*) would produce rotations 0.5 at A and B as shown in straight dotted lines in Fig. 12*b*, making $EE' = 0.25L$ and $y_m = 0.5 \times 0.25L = 0.125L$; however due to restraint at A and B tangents are turned back by 0.259 at A and 0.343 at B making $f = 0.157L$ and $f' = 0.241L$; according to Table II, then, $y_m = L(1 + 2 \times 0.157 + 2 \times 0.241)/24 = 0.0748L$.

Stress computation: $M_E = Py_m = 0.0748PL$.

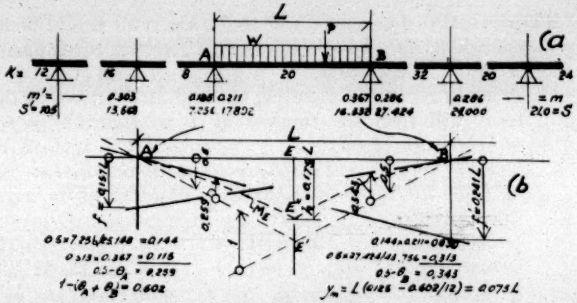

FIG. 12. Exact analysis of M_E in center of span AB of continuous beam of "infinite" number of spans.

Example 4: Irregular portal frame, fixed at A and D, as shown in Fig. 13, but loaded by horizontal pressure P at B only.

Wanted: Bending moments at A, B, C and D and thrusts at A and D, treated as straight beam.

Analysis: The lines in Fig. 14 show, (1) the k values, (2) the m (equation 1), (3) the S (equation 3), (4) and (5) the rotations at B and C for unit displacement (Fig. 7*b*) at A and D respectively, (6) the total rotations at B and C as the sum of (4) and (5); (7) the $m = f/f'$ and $m' = f'/f$ of 6), (8) the $\bar{e}$ and $\bar{e}'$ (equation 9) of (6), (9) the portion of the horizontal pressure P absorbed by $T_A = 0.7705/1.7705 = 0.4352$ and $T_D/P = 1/1.7705 = 0.5648$, being $\bar{e}_2\bar{e}_3'/\bar{e}_2'\bar{e}_1' = 0.7705$ (Fig. 7*b*), (10)

the bending moments $M_A = T_A\bar{e}_1$, $M_B = T_A\bar{e}_1'$, $M_C = T_D\bar{e}_3'$ and $M_D = T_D\bar{e}_3$.

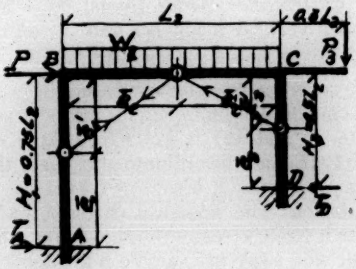

FIG. 13. Irregular portal frame *analyzed in Figs. 11 and 12.*

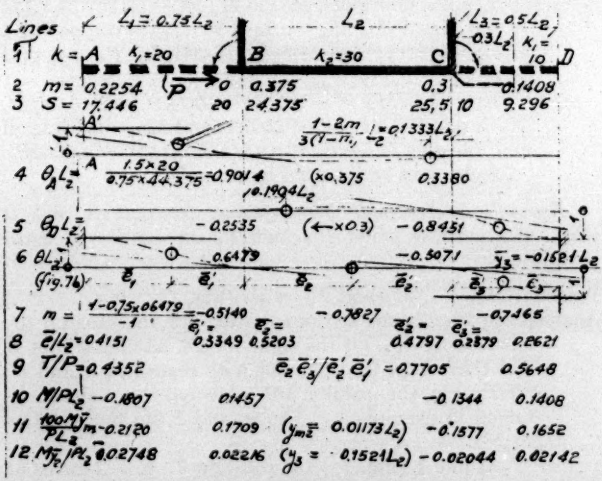

FIG. 14. Portal frame (Fig. 10) analyzed as straight continuous beam for side sway "1" or horizontal load P.

Stress computation: $M_A/PL_2 = -0.4352 \times 0.4151 = -0.1807$, $M_B/PL_2 = 0.4352 \times 0.3349 = 0.1457$, $M_C = -0.5648 \times 0.2379 = -0.1344$ and $M_D = 0.5648 \times 0.2621$

$= 0.1408$. Further (9) $T_A/P = 0.4352$ and $T_D/P = 0.5648$.

Example 5: Irregular portal frame, fixed at A and D (Fig. 13) under vertical loads W_2 and P_3 on cantilever at $0.3L_2$ beyond C.

Wanted: Side sway stresses (for correcting in example 6).

Analysis: The medium ordinate $\bar{y}_{m2}$ for unit side sway (Fig. 14, lines 6, 11 and 12) is $\bar{y}_{m2} = L_2(0.6479 - 0.5071)/12 = 0.01173L_2$ and the ordinate at $0.3L_2$ of the cantilever at C is $y_3 = 0.3L_2 \times 0.5071 = 0.1521L_2$.

Stress computation: The side sway stresses, to be added or deducted from stresses obtained for the structure without side sway in example 6 are for loads W_2 and P_3 (Fig. 13) as shown in lines 11 and 12 (Fig. 14).

For W_2, multiplied by 100 $M_A = -0.1807 \times 1.173 = -0.2120W_2L_2$, $M_B = 0.1457 \times 1.173 = 0.1709W_2L_2$, $M_C = -0.1577$ and $M_D = 0.1652W_2L_2$, $T_A = 0.4352 \times 1.173^2 = 0.0051W_2$.

For P_3: $M_A = -0.1807 \times 0.1521 = -0.02748$, $M_B = 0.02216$, $M_C = -0.02044$ and $M_D = 0.02142$ all multiplied by P_3L_2, $T_A = 0.4352 \times 0.1521 = 0.0662P_3$.

Signs may be inverted in example 6.

Example 6: Irregular portal frame, fixed at A and D (Fig. 13), vertical loads W_2 and P_3 on cantilever $0.3L_2$ beyond C.

Wanted: M_A, M_B, M_C and $T_A = T_D$ (Fig. 15).

Analysis: In Fig. 15 the stresses wanted are analyzed on the left side for W_2, on the right for P_3, assuming that B and C are kept from moving with regard to A and D; afterwards the results are corrected by values obtained in example 5. The m and S are the same as in Fig. 14.

At line 1 (Fig. 15) the rotation $\theta_A = -1$ turns at B by $-m_1\theta_A = 0.2254$, making (line 2) $y_{m2} = 0.2254 \times 1.375L_2/12 = 0.02583L_2$ and $y_3 = -0.2254 \times 0.375 \times 0.3L_2 = -0.02536L_2$. In line 3 are shown the M_A (Fig. 14, lines 11 and 12) due to side sway of BC with respect to AB and in line 4 the results obtained, $y_m = 0.02371L_2$ and $y_3 = -0.05278L_2$.

Figure 15 shows the analysis of M_B on lines 5 to 8, M_C on lines 9 to 12 and T_A on lines 13 to 16.

Stress computation: With the ordinates obtained in the analysis (Fig. 15) the stresses wanted are:

$$M_A = 0.02371W_2L_2 - 0.05278P_3L_2,$$
$$M_B = -0.04993W_2L_2 + 0.07286P_3L_2,$$
$$M_C = -0.03210W_2L_2 - 0.2359P_3L_2,$$
$$T_A = 0.09818W_2 - 0.1676P_3.$$

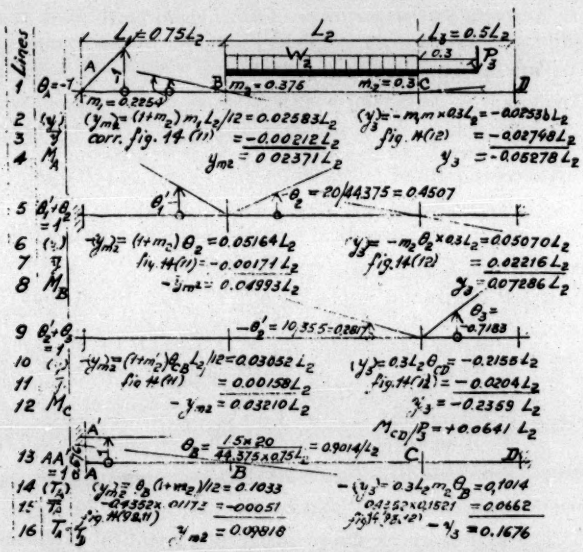

FIG. 15. Portal frame (Fig. 13) analyzed as straight continuous beams for bending moments and hor. thrust (y_{m2}) and (y_3) for structure without side sway.

It should be noted that ordinates y on top of the axis $ABCD$ have a different sign from those below $ABCD$.

Example 7: Oblique frame ABC, hinged at A and C (Fig. 8) loaded by vertical load V_1 and horizontal load W_2. Wanted: Bending moment M_B.

Analysis: According to explanations given above we analyze the vertical and horizontal projection of the triangular frame, rotating by "1" the continuous beam $AB''C$ at B'' and the double beam $B''BB''$ (Fig. 8) at B. In both projections is $\theta_1' = S_2/S = 0.4$ and $\theta_2 = S_1'/S = 0.6$ and therefore the medium ordinates for AB'': $y_{m1} = 0.4L_1/8 = 0.05L_1$ and for BB'': $y_{m2} = 0.6H/8 = 0.075H$.

Stress computation: $-M_B = 0.05V_1L_1 + 0.075W_2H$.

Example 8: Oblique frame ABC, hinged at A fixed at C (Fig. 9), loaded by vertical load V_1 and horizontal load W_2.

Wanted: Bending moment M_B.

Analysis: $S_1' = 0.75k_1 = 18$; $S_2 = k_2 = 16$. $\theta_1' = S_2/S = 0.471$, $\theta_2 = S_1'/S = 0.529$. $-y_{m1} = \theta_1'L_1/8 = 0.059L_1$, $-y_{m2} = 0.529L_2/12 = 0.0441H$.

Stress computation: $-M_B = 0.059L_1V_1 + 0.0441HW_2$.

Example 9: Oblique frame ABC, hinged at A and C (Fig. 10), loaded by vertical load V_1 and horizontal load W_2.

Wanted: Horizontal thrust T_A and T_C.

Analysis: Horizontal displacement "1" (Fig. 10) at A or C produces at B: $\theta_1' = S_2/SH = 0.4/H$ and $\theta_2 = S_1'/SH = 0.6/H$, therefore $y_{m1} = L_1L_2/2LH + 0.4L_1/8H = 0.5L_1L_2/LH + 0.05L_1/H$ and $-y_{m2} = L_1/2L + 0.6H/8H = 0.5L_1L + 0.075$.

Stress computation: $T_A = V_1(0.5L_1L_2/LH + 0.05L_1/H) - W_2(0.5L_1/L + 0.075)$. $T_D = V_1(0.5L_1L_2/LH + 0.05L_1/H) - W_2 \times 0.425$.

CONCLUSION.

The same way as shown above other statical problems might be reduced especially arches; the natural simplification attained by the correct design in relation to the current abstract methods becomes most marked in continuous beams and frames. It is a further convenience, that the geometrical results once obtained for any one structure are good for any load or change of load and that the designers' attention is held at the structure and not diverted by voluminous abstract calculations.

NOTES FROM THE NATIONAL BUREAU OF STANDARDS.*

EXCESSIVE EXPANSION OF DENTAL AMALGAMS.

A research has just been completed by I. C. Schoonover and associates which explains the reason for excessive expansion of dental amalgams—a defect which has so frequently puzzled the dentist.

Dental alloy when amalgamated should expand slightly (about 8 microns per centimeter) when it sets and, after 10 to 20 hours, should show no further expansion. In some instances, however, specimens have been observed to expand over 200 microns in 30 days.

Records show that these excessive expansions occur only with alloys containing zinc and then only if the alloy is contaminated with salt solutions (e.g., NaCl) obtained from the perspiration of the palms. Salt added intentionally during amalgamation produces similar expansion. Gas pockets were found in such expanding amalgams, in some instances the pockets having broken through the surface of the amalgam as blisters. The gas has been identified as hydrogen. The pressure in these pockets builds up and, after 24 to 48 hours, is sufficient to cause the specimen to expand or "spheroid."

Non-zinc alloys do not show this abnormal expansion nor do zinc-containing alloys if they are amalgamated in a clean mortar or a mechanical amalgamator and are not contaminated by handling.

These results should not be construed as an endorsement of zinc or non-zinc alloys but should rather be accepted as a justification for the utmost care and cleanliness in handling these alloys.

The complete report is being prepared for publication in a future issue of the *Journal of the American Dental Association.*

* Communicated by the Director.

COPPER TANKS IN INDUSTRY.

Copper tanks, vats, and containers of various shapes and sizes are widely used in industry. Some of these are of very large size. Although the necessary information is available on which to base recommendations as to the chemical uses to which such containers may safely be put, information which will serve as a guide in their mechanical construction and assembly is woefully scant. Rule of thumb methods must be used or dependence placed on principles established for corresponding steel construction. To meet the need for this information, the Copper and Brass Research Association, a technical organization representing the copper and brass industry, has just established a new industrial research project in the Bureau's Division of Metallurgy dealing with this subject. A good deal of testing work will be required to assemble the necessary data on the basic mechanical properties of flat copper in different gages and tempers. Another subject that will require study is the determination of the strength and general durability of the various joints used in such construction work. Finally, all the data and information must be assembled and correlated with the basic engineering principles of construction so as to be readily usable by designers and builders.

STRESSES IN RIGID FRAMES.

Engineers have for some time been interested in the distribution of stress in the knees of rigid frames. Numerous tests have been made to determine this distribution; but tests alone are of limited value unless they permit generalization, that is, unless it is possible to predict the behavior of other similar but different structures on the basis of the tests. One method of generalizing consists in analyzing tests by well-founded theory which is not restricted by the particular dimensions of a test specimen nor the conditions of loading. If the limitations of the theory are kept in mind, wide generalization is often permissible.

In *J. Research N.B.S.* 21, 269 (September, 1938), RP1130, an analysis was given of the stresses in a rigid frame. A

further development of this work by William R. Osgood is reported in the *Journal of Research* for November (RP1431). The analysis here presented is more accurate than the earlier one.

WATER PERMEABILITY OF MASONRY WALLS.

The effects of weathering exposure on the water permeability of about 100 small masonry walls has been observed by exposing the walls outdoors at Washington for a period of two or three years. The permeability of the walls was determined both before and after outdoor storage by testing them under conditions which produced the effects of a wind-driven rain. Some of the walls that had leaked badly when first constructed had been waterproofed before being exposed outdoors and the durability of the waterproofing treatments was also observed.

The information obtained through these tests has now been published as Building Materials and Structures Report BMS76. Copies are obtainable from the Superintendent of Documents, Government Printing Office, Washington, D. C., for 15 cents.

The weathering exposure did not have an important effect on the permeability of all-brick or brick-faced walls that were 8 inches or more in thickness. Stucco-faced walls with tile backings were more permeable after exposure than before, but their performances were still satisfactory. Cutting out and repointing the face joints was the most effective and durable of the waterproofing treatments.

The application of a portland cement grout, thinly applied with a brush to the joints of a brick wall, was still effective after three years' exposure although the permeability of the walls, so treated, had increased. Both oil and cement-water paints were still effective as waterproofings after 3 years although showing evidence of wear.

THE PLANOFLEX FOR EVALUATING THE PLIABILITY OF FABRICS.

The "hand" or "feel" of fabrics and its evaluation are important alike to the finisher, manufacturer, and consumer.

With the advent of many new fibers and textile finishing agents the tactual method of determining hand has become inadequate. Accordingly, the American Society for Testing Materials and the Bureau, with the support of the textile industry are developing quantitative objective test methods and instruments. The first of these is the "Planoflex" which is used to evaluate the pliability of a fabric, one of the chief components of hand. It is described in a report (RP1434) by Edwin C. Dreby, which will appear in the November number of the *Journal of Research.*

The Planoflex is used for measuring the extent to which a fabric can be distorted in its own plane without producing wrinkles on its surface. Results of measurements on a series of cotton percales showed an 88 per cent. correlation with their tactual pliability ratings. The Planoflex appeared to be considerably more sensitive than the experts in the range of pliability exhibited by these fabrics.

The Planoflex was shown to be as good as, or better than, the Schiefer Flexometer and the Peirce Hanging-Heart Loop methods for evaluating pliability with respect to the extent of the correlation between measured values and tactual pliability ratings, sensitivity to small differences in pliability, and ease of operation.

The Planoflex will give measurable values on all woven fabrics except those that are so heavily starched that the filling and warp threads are cemented together. It is not applicable to knit fabrics since they are so easily stretched and distorted that they cannot be mounted in the instrument. Since the construction of a fabric affects to a large extent its angle of distortion, measurements with the Planoflex are only comparable for similarly constructed fabrics. The sensitiveness of the Planoflex to slight differences in pliability makes it a valuable tool for comparing quantitatively the effects of different finishing agents. Its simplicity of operation and rugged construction make it particularly adapted to routine testing and control work during the processing of fabrics.

THE FRANKLIN INSTITUTE.

STATED MONTHLY MEETING, 8:15 P.M. WEDNESDAY, NOVEMBER 19, 1941.

The regular monthly meeting of The Franklin Institute was called to order at 8:15 o'clock by the President, Mr. Philip C. Staples.

The minutes of the previous meeting were approved as printed in the November issue of the JOURNAL.

The Secretary reported the following additions to the membership during the month of October:

Active	16
Associate	21
Student	35
Total	72

The Secretary made the announcement that the Christmas Week Lectures for Young People, on the James Mapes Dodge Foundation, this year will be given on Monday, Tuesday and Wednesday, December 29, 30, 31. The lecturer will be Dr. Hubert Newcombe Alyea, who has chosen as his subject "Man-Made Molecules."

Tickets for the above lectures and membership in The Franklin Institute were suggested by the Secretary as holiday mementoes for friends.

Members were reminded of the subscription dinner on December first, when the Vermilye Medal will be awarded to W. S. Knudsen, Director General, Office of Production Management, Washington, D. C. This award is given by the Institute every two years, "for outstanding contribution in the field of industrial management."

The Chairman then introduced Dr. Ellice McDonald, Director of the Biochemical Research Foundation, who presented his annual report of the progress of the laboratories to The Franklin Institute.

Dr. McDonald spoke of the year's work, explained the arrangement of the new laboratories and apparatus, and accompanied his talk by descriptive lantern slides.

The meeting was adjourned by a rising vote of thanks to the lecturer, and an invitation from the Chairman to the audience to inspect a recent gift to the Institute—a lap desk which is recorded as having been the property of Benjamin Franklin. This interesting and valuable piece was donated to the Institute by Stanley G. Flagg, in memory of his uncle, John T. Windrim. Mr. Windrim, a long-time member of the Institute, was architect of the new building, opened in 1933, "in honor of Benjamin Franklin."

HENRY BUTLER ALLEN,
Secretary.

LIBRARY NOTES.

Photostat Service. Photostat prints of any material in the collections can be supplied on request. Orders received in the morning are filled the same day. The average cost for a print 9 × 14 inches is thirty-five cents.

The library and reading room are open on Mondays, Tuesdays, Fridays and Saturdays from nine o'clock A.M. until five o'clock P.M., Wednesdays and Thursdays from two until ten o'clock P.M.

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COMMITTEE ON SCIENCE AND THE ARTS.

(*Abstract of Proceedings of Stated Meeting held Wednesday, November 12, 1941*)

HALL OF THE COMMITTEE,
PHILADELPHIA, NOVEMBER 12, 1941.

MR. COLEMAN SELLERS, 3RD, *in the Chair.*

The following report was presented for final action:

No. 3085: Ingersoll "Koolshade" Sun Screen.

This report recommended the award of a Certificate of Merit to John J. Grebe, of Midland, Michigan, "In consideration of his invention of a metal screen which effectively reduces solar radiation entering through windows."

JOHN FRAZER,
Secretary to Committee.

NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION.

The Ultracentrifugation of Rat Plasma Phosphatase.—ELMER O. KRAEMER, LEOPOLD WEIL, EDWARD B. SANIGAR AND MARY THOMAS ALLEN. The principal protein constituents of blood have been rather thoroughly investigated in their native state by ultracentrifugal and related methods in order to obtain information regarding their molecular sizes, but the minor protein constituents, such as the enzymes, have been largely neglected. Since the concentrations of the enzymes in blood are quite low and since most of them have no sufficiently characteristic light absorption spectra, it is not possible to make observations on their diffusion or sedimentation in the ultracentrifuge with the optical methods generally used (T. Svedberg and K. O. Pedersen, "The Ultracentrifuge," Oxford University Press, 1940). For such conditions, however, Tiselius, Pedersen and Svedberg (*Nature*, **140**: 848, 1937) proposed the use of a cell with a permeable partition perpendicular to the direction of sedimentation, which prevents mixing of the entire solution when the ultracentrifuge is stopped. By chemical analysis of the solutions in the upper and lower compartments, and the application of a mode of calculation similar to that employed in the Hittorf transport-number method, Tiselius *et al.* have shown how sedimentation constants may be calculated from analytical data alone. This method is now being applied in this Laboratory to the study of blood phosphatase.

Rat blood was taken from the heart, placed in oxalated tubes, and the plasma separated from the blood cells in the usual manner. The rats were not starved because of the decrease in plasma phosphatase activity caused by starvation (L. Weil and M. A. Russell, *Journal of Biological Chemistry*, **136**: 9, 1940). All the enzymatic determinations were carried out on the plasma diluted to a final concentration of 1 : 10 with 0.9% NaCl solution. To 7 cmm. of plasma (1 : 10 dilution), 7 cmm. of HCl-veronal buffer of pH 9, containing

0.015 M $MgCl_2$, and 7 cmm. of 0.1 M Na-β-glycerophosphate were added and the mixture incubated for 2 hours at 37° C., after which the liberated phosphate was determined by the colorimetric micro-method previously described (L. Weil, *Journal of Biological Chemistry*, **138**: 375, 1941). The amount of enzymatic decomposition of the substrate was kept below 0.60 microgram of phosphorus during the two hours of incubation, because under these conditions the reaction obeyed the unimolecular reaction law and the reaction constants could be considered as proportional to the phosphatase concentrations.

Ultracentrifugation was done in a Svedberg oil-turbine machine at a speed of 39,000 ± 1,000 r.p.m. and a cell temperature of approximately 21° C., the running time in each case being 90 minutes. The cell used was 18 mm. in length along the optical path and had Lucite windows. These windows were not satisfactory for accurate optical analysis with so long a cell, but sufficed for optical control of the experiments. Photographs, using the Lamm scale method, were taken in order to follow the sedimentation of the principal protein constituents. After each centrifugation the contents of the upper and lower compartments of the cell were individually mixed by careful stirring, and were separately withdrawn for analysis.

The sedimentation constants, s, were calculated from the contents of both upper and lower compartments, using for the upper compartment the equation

$$s_u = -\frac{1}{2\omega^2 t} \ln\left[\left(\frac{x_0}{x_i}\right)^2 (1 - F_u) + F_u\right]$$

and for the lower compartment,

$$s_b = -\frac{1}{2\omega^2 t} \ln\left[1 + C - CF_b\right],$$

where F_u and F_b are the fractional amounts of phosphatase, relative to the original amounts, in the upper and lower compartments, respectively;

$$C = \left[\frac{x_b^2 - x_i^2 - 2\beta x_i}{x_i^2}\right],$$

where x_b and x_t are the distances of the bottom of the cell and the top of the partition from the center of rotation, and $\beta = (1 - r)d$ where r is the relative area of the holes in the partition and d is its thickness; ω is the angular velocity; x_0 is the distance from the meniscus to the center of rotation and t is the time of centrifugation.

The results of some experiments with rat plasma, undiluted and diluted with various quantities of 0.9% NaCl solution, are shown below. The above sedimentation con-

Plasma Concn.	F_u.	F_b.	$s_u \times 10^{13}$ sec.	$s_b \times 10^{13}$ sec.	Total Recovery of Phosphatase %	Position of Phosphatase Boundary x_p.	$x_p - x_0$ cm.
$p/1$	0.528	1.198	4.26	4.40	98.9	6.140	0.231
$p/2$	0.495	1.217	4.71	4.84	98.5	6.135	0.253
$p/5$	0.511	1.210	5.16	4.78	97.2	6.076	0.256

stants s_u and s_b are not corrected for the viscosity of the medium nor to a basis of sedimentation in water at 20° C., and they are not, therefore, directly comparable with values of s_{20} for either albumin or globulin. They do, however, tend to increase with decreasing plasma concentration, which is usual for the sedimentation constants of proteins. The values for recovered phosphatase represent the sum of the amounts of phosphatase found after centrifugation in the upper and lower compartments compared with the amount of phosphatase originally present in the same total volume. The calculated positions, x_p, of the phosphatase boundaries, relative to the positions of the principal protein boundaries at the completion of two of the centrifugations are shown by arrows on the sedimentation diagrams of Fig. 1. A sedimentation diagram for the undiluted plasma could not be obtained because the concentrated solution together with the long optical path through the cell caused a *schlieren* band which precluded usable scale photographs.

From the data presented it will be seen that the phosphatase sediments with, or quite near to, the principal protein constituents and, specifically, near the globulins. On the other hand, C. Cattaneo (*Enzymologia*, **2**: 356, 1938), found by chemical fractionation of human blood serum that the

phosphatase activity was associated exclusively with the albumin fraction. In our experiments the resolving power attained was insufficient to permit the separation of the albumin and globulin, or of the phosphatase and the principal proteins. To establish this point it will be necessary to use a

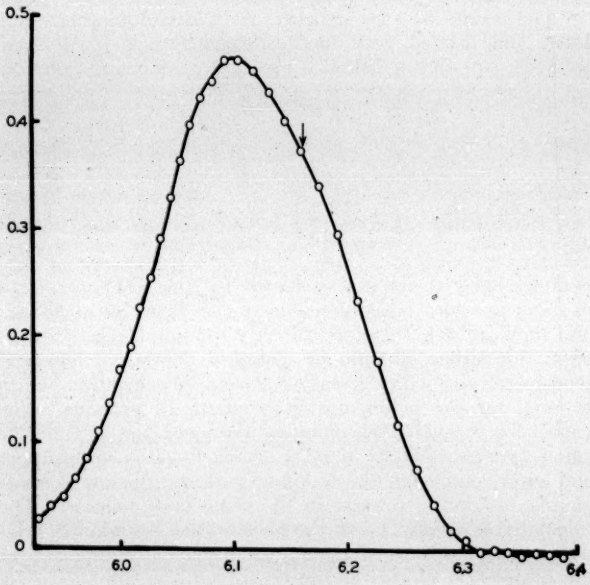

FIG. 1*a*. Plasma/2.

cell having the partition nearer the bottom, in order to increase the resolving power. Such experiments are planned, as well as experiments at lower phosphatase concentrations.

It is interesting to compare our results with recent observations by Kabat (*Science*, **93**: 43, 1941) who found by high speed centrifuging (27,000 r.p.m. for one hour) that fresh kidney extracts contained phosphatase in a rapidly sedimenting form, the phosphatase presumably being attached to

large particles referred to as the "heavy fraction" by Kabat. By autolysis, however, the phosphatase was liberated from the large particles and was no longer sedimentable under the conditions Kabat used. Since blood phosphatase would also be non-sedimentable under these conditions, it will be neces-

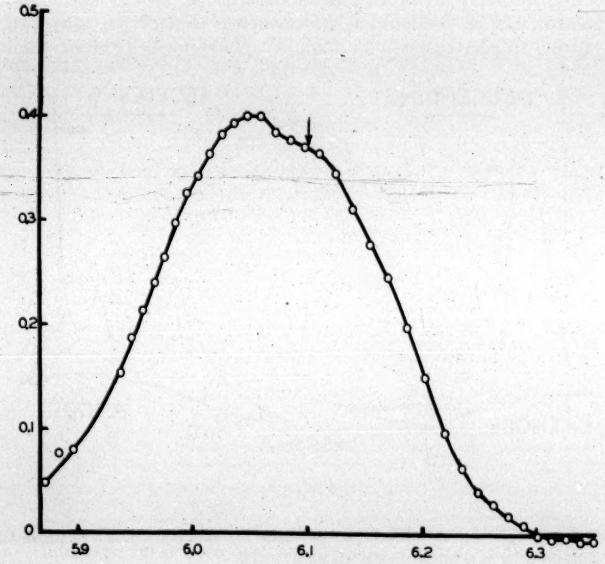

FIG. 1b. Plasma/5.

FIG. 1. Sedimentation Diagrams of Rat Plasma after Ultracentrifugation for 90 minutes.

Abscissas: Distance in cm. from center of rotation.
Ordinates: Scale-line displacement in mm.

sary to extend our experiments to autolyzed kidney extracts in order to determine whether Kabat's "liberated phosphatase" is similar in sedimentation behavior (and therefore particle size) to blood phosphatase.

Electrophoretic Evidence for Complex Formation in Casein.—LAURA E. KREJCI, ROBERT K. JENNINGS, AND LOUIS DESPAIN SMITH. Electrophoresis in the apparatus of Tiselius (*Transactions of the Faraday Society*, **33**: 524, 1937), using the Toepler *schlieren* method for observation of the boundaries, has shown that acid-alcohol soluble casein contains three constituents, which have been designated alpha, beta, and gamma casein (Mellander, *Biochemische Zeitschrift*, **300**: 240, 1939). Alpha casein has a higher phosphorus content than

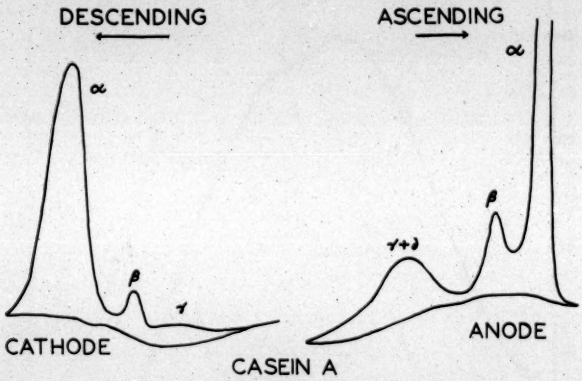

FIG. 1.

gamma casein, and its comparatively high mobility has been ascribed to its phosphoric acid groups.

During an investigation of antibody inactivation in the presence of casein (Krejci, Jennings, and Smith, communicated to the *Journal of Immunology*) the electrophoresis of casein was followed by means of the *schlieren* scanning method of Longsworth (*Journal of the American Chemical Society*, **61**: 529, 1939), which gives information concerning the relative concentrations of the constituents. Two samples of acid-alcohol soluble casein, A and B, were prepared at different times by fractionation of commercial casein as described by Svedberg, Carpenter, and Carpenter (*Journal of the American*

Chemical Society, **52**: 241, 1930). Approximately fifteen grams of casein were extracted at 37° C. for periods of two hours each with two consecutive portions of 70 per cent. ethyl alcohol to which 2 ml. of normal HCl had been added. The second extract was carefully neutralized with NaOH; the finely divided precipitate was recovered with a Sharples supercentrifuge, dissolved in buffered sodium chloride solution, and dialyzed first against running tap water, then against buffered sodium chloride solution. For electrophoresis, the solutions were dialyzed against large volumes of a buffer solution of pH 7 containing 0.15 M NaCl and 0.01 M total phosphates.

TABLE I.

Electrophoretic Mobilities of Acid-Alcohol Soluble Casein. Buffer: pH 7, 0.01 M phosphate, 0.15 M NaCl. T: 0° C.

Experiment.	Lot.	Total Concentration (Computed from Total Area of Electrophoretic Diagram and Expressed as Refractive Increment $\times 10^4$).	$u \times 10^5$ (in cm.^{2} volt^{-1} sec.$^{-1}$)					
			Descending Boundaries (Cathode Arm).			Ascending Boundaries (Anode Arm).		
			α	β	γ	α	β	γ
1	A	258	−5.5	−2.4	+0.7	−5.4	−3.4	+0.1
2	B	206	−5.4	−2.5	−0.4	−5.6	−3.4	−0.2
3	B	159	−4.7[a]	−2.1[a]	−0.3[a]	−5.0[a]	−2.7[a]	0.0[a]
4	B	159	−5.1	−2.4	−0.4	−5.2	−3.2	−0.3

[a] The mobilities for this run were all low (by about 0.4), probably as the result of a slow leak at one of the rubber joints, which caused movement of the solution as a whole through the U-tube in the direction opposite to that of migration.

The electrophoretic patterns in the two arms of the U-tube were not symmetrical (see Fig. 1). The mobilities, calculated from the lateral displacement of the peaks, are listed in Table 1; and the relative concentrations of the constituents, computed from the areas beneath the curves, are tabulated in Table 2. Normally the concentrations in the anode arm are smaller than those in the cathode arm because of dilution at the delta boundary (Longsworth and MacInnes, *Journal of the American Chemical Society*, **62**: 705, 1940); the concentration of beta casein, however, was greater in the anode arm than in the cathode arm, and the mobility was higher.

Furthermore, in the anode arm the reduction in the concentration of the alpha casein was greater than that predicted by the magnitude of the delta effect. The lack of symmetry is an indication of the formation of unstable complexes (Longsworth, Cannan, and MacInnes, *Journal of the American Chemical Society*, **62**: 2580, 1940; Chargaff, Ziff, and Moore, *Journal of Biological Chemistry*, **139**: 383, 1941), in this instance, complexes of beta casein with alpha casein. In the anode arm, where the beta casein boundary migrated in a medium containing alpha casein, the concentration in the

TABLE II.

Percentage Composition of Casein Solutions.

Experiment.	Lot.	Total Concentration (Computed from Total Area of Electrophoretic Diagram and Expressed as Refractive Increment $\times 10^5$).	Relative Concentrations.					
			Descending Boundaries (Cathode Arm).			Ascending Boundaries (Anode Arm).		
			α	β	γ	α	β	γ
1	A	258	83%	9%	8%	65%	16%	19%
2	B	206	80%	9%	11%	63%	15%	22%
3	B	159	77%	11%	13%	66%	16%	18%
4	B	159	77%	10%	13%	64%	16%	20%

beta boundary increased at the expense of the alpha casein concentration, and the mobility was enhanced as the result of the complex formation by beta casein with this more rapidly migrating constituent. In the cathode arm, where the beta casein boundary migrated in a medium free from alpha casein, the complex was disrupted above the receding alpha casein boundary, and the concentration and mobility were those of uncombined beta casein.

Within the range studied the total concentration of casein appeared to have little influence on the extent of complex formation or on the mobility of the complex. The magnitude of the delta boundary, which could not be determined experimentally because of lack of resolution from the gamma casein boundary, was estimated from the amount of dilution observed in the anode arm for the sum alpha casein plus beta casein. When the concentrations of alpha, beta, and gamma casein

in the anode arm were corrected for this dilution, the following values were obtained:

Total Concentration	Anode Arm (Calculated)			Cathode Arm (Measured)		
	α	β	γ	α	β	γ
258	74%	18%	8%	83%	9%	8%
206	72%	17%	11%	80%	9%	11%
159	70%	17%	13%	76%	11%	13%
159	70%	17%	13%	77%	10%	13%

Assuming equal refractivities, it would appear that the beta casein combined with somewhat less than its own weight of alpha casein. The figures show that with decreasing total concentration there was a downward trend for the concentrations of alpha casein and an upward trend for the concentration of gamma casein. These differences are small, and they may have resulted from chance errors; they would, if substantiated, be an indication of an additional more complicated type of interaction in which all the constituents of casein participate.

Ultracentrifugal sedimentation measurements (Philpot and Philpot, *Proceedings of the Royal Society*, **B 127**: 21, 1939) have shown that in the absence of calcium ions casein has a sedimentation constant of about 6. The addition of increasing amounts of calcium (0.003–0.020 moles per liter) leads to molecular association and causes a gradual increase of the sedimentation constant up to about 10.4; it appears that equilibrium between single and associated molecules is established so rapidly with respect to the rate of sedimentation that the various species sediment as a single constituent at an intermediate rate. The calcium content of the stock solution of casein B, from which the solutions used for experiments 2, 3, and 4 of the present investigation were prepared by dilution, was 0.0003 moles per liter. Since in addition the solutions were dialyzed against large volumes of phosphate buffer solution, the equilibrium between the single and the associated molecules resulting from the presence of calcium ions should presumably have been shifted completely in the direction of the unassociated molecules.

The demonstration by means of electrophoresis that about one-sixth of the casein consists of complexes of approximately twice the weight of the unassociated molecule is not, however, incompatible with the ultracentrifuge results. This proportion of a stable complex of twice the weight would, except in centrifugal fields of extreme intensity, be evident only as a distortion of the sedimentation diagrams for the unassociated casein. An unstable complex would be even more difficult to detect. It would cause an increase of the sedimentation constant but the magnitude of the increase would be near the limit of experimental error.

The ultracentrifuge results, therefore, may be said to be consistent with the electrophoretic results: the three constituents revealed by the electrophoresis of casein are not mutually independent, but beta casein forms unstable complexes with alpha casein.

BOOK REVIEWS.

A TEXTBOOK IN ELECTRICITY AND MAGNETISM, by Harry C. Kelly, 346 pages, illustrations, 16 × 24 cms., New York, John Wiley & Sons, Inc., 1941. Price $3.75.

Modern textbooks on this subject are of importance primarily for several reasons. Because of the rapid progression of scientific knowledge generally the text must reflect the latest concepts of underlying principles. It must fit in with educational programs, that is to say, its scope and treatment of details must be sufficient in extent. It must have appeal to the student and arouse curiosity. Its method of presentation must be such as to make lasting impressions, yet the work must not be fatiguing.

These and other features are contained in the book at hand. It follows the study of mechanics and heat, and proceeds with mechanics under the topics of gravitation and gravitation potential to introduce, through energy relations, electricity. Following this there are in order studies on electrostatus, electric current, electrolytes, magnetostatics, cathode and canal rays, induced E. M. F., electronics, the new concept of the atom, alternating currents, and electromagnetic waves. The trend throughout is logical and clear. Problems at the end of chapters reveal keen insight to the situation of the student and although the treatment at times is somewhat rigorous, everywhere there is a sympathetic attitude toward and understanding of the needs and reactions of the student. The book is of value as a text for classroom use.

R. H. OPPERMANN.

AN INTRODUCTION TO PHYSICAL STATISTICS, by Robert B. Lindsay, 300 pages, illustrations, 16 × 24 cms., New York, John Wiley & Sons, Inc. 1941. Price $3.75.

Statistics for some time has been firmly established as a practical means for the solution of physical theories and problems. In fact, the use of the methods of statistics have more than once provided "a way out" for physicists who have gotton into situations where other means have failed. Today a knowledge of physical statistics is essential as a tool not only to provide "a way out," but, and far more important, as a means of describing physical phenomena not describable by other means, as for instance the association of a number with time on a probability basis, or when the number is an average.

This book is planned, according to the author, to fit into the niche where a student is about to embark on graduate study and who wishes a thorough but not too lengthy introduction to the method of statistical physics. It begins with the study of elementary probability and statistics, and a review of thermodynamic principles in so far as to make it possible to give a coherent description of a large number of physical and chemical phenomena. This is the foundation for the work which follows—to show how the statistical point of view provides an interpretation of thermodynamics in terms of the atomic constitution of matter.

Of the ways of doing this, there are considered the classical statistics of Maxwell and Boltzmann, the classical kinetic theory where it is assumed that the student is acquainted with the fundamentals of the kinetic theory of gases, the statistical mechanics of Gibbs, the statistical mechanics of Darwin and Fowler which is the more recently developed method and which abandoned the concept of Gibbs, and the application of quantum mechanical ideas to statistics and statistical distribution. The concluding part of the book is devoted to discussions on specific heats of gases and solids, the quantum statistical theory of electrical and thermal properties of metals, and the emission of electrons from surfaces.

The book adequately fulfills its purpose. It is a text that satisfies a definite need in preparation for advanced work.

R. H. OPPERMANN.

THE PHOTOCHEMISTRY OF GASES, by William A. Noyes, Jr., and Philip A. Leighton, 475 pages, illustrations, 16 × 24 cms., New York, Reinhold Publishing Corporation. Price $10.00.

No better subject than this can be presented for discussion for the purposes of assisting workers in various fields in which knowledge of photochemistry is useful, and of pointing out directions for further investigation. Photochemistry is one of the new sciences that bridges the older sciences of chemistry and physics. Photochemistry of gases is only a part of the main subject, but a most important part and while much progress has been made in theorizing and experimentation to the end of greater knowledge, there remains much work to be done in definitely establishing this knowledge as well as forging ahead into what is now unknown.

This book, a monograph of the American Chemical Society, reviews a tremendous amount of data from a wide variety of sources. As its title indicates, it is restricted to gases. In this phase, further limitations are imposed depending on the optical properties of substances which can be used to provide windows for reaction vessels, generally in wave-lengths from the red (about 7×10^{-5}) down to 1.2×10^{-5} cms. After discussing the experimental technique of photochemistry, the treatment, here presented, makes a survey of spectroscopy from the viewpoint of the experimentalist, which is followed by the interpretation and significance of the overall or gross quantum yield of a photochemical reaction. This leads into consideration of the kinetics of thermal reactions. Photochemical reactions resulting from absorption by atoms is the next topic, and reactions following absorption by diatomic molecules for which quantitative data are available. In this latter topic some critical discussion is given and there is revealed doubtful points particularly in cases in which mechanisms are not established.

The last part of the book is devoted to reactions following absorption by polyatomic molecules. The applications of the methods which may be used to determine the nature of primary dissociation are illustrated by specific cases and there is a discussion of specific reactions of polyatomic molecules confined almost entirely to those reactions in the gas phase for which quantitative data have been obtained and for which definite information concerning the mechanism can be deduced. Here again a critical attitude is adopted, discrepancies indicated, and in some cases suggestions made for the purpose of more certain interpretations.

Appendices contain data of photochemistry in tabular form. Innumerable references are cited throughout the work and there is a bibliography and subject index at the end.

An exhaustive search through the literature, and careful selection and classification of pertinent data and information are the foundations of this work. Expert knowledge of the subject; a keen sense of the art of presentation; the ability to analyze, interpret, translate and correlate the data all contribute to make the work a valuable one to chemists, physicists and students.

R. H. OPPERMANN.

WORKING STRESSES, by Joseph Marin, 41 pages, illustrations, 14 × 22 cms. New Brunswick, Rutgers University Press, 1940. Price $1.00.

Within the thirty-nine pages of this book there is discussed the working stresses for static and fluctuating loads, and creep conditions. The first of the above topics discusses in point simple tension, static combined stresses and the working stresses for brittle materials for combined static stresses. Under fluctuating loads, axial and combined stresses is covered and with regard to creep conditions cases of simple tension, pure bending and torsion are considered. Some twenty-one curves and diagrams supplement the treatment.

R. H. OPPERMANN.

PARTIAL DIFFERENTIAL EQUATIONS, by Frederic H. Miller, 259 pages, 16 × 25 cms., illustrations. John Wiley & Sons, New York, 1941. Price $3.00.

The successful teaching of mathematics depends to a great extent on the completeness of coverage. Often insufficient coverage is not realized until mathematics as a tool for the solution of practical problems is needed and the very elements of the subject must come into use. The subject of this book is supplementary to the usual course in differential equations for the purpose of giving completeness of coverage, particularly the elements.

The text begins with two chapters of preliminary review work, the first devoted to a summary of the methods of solving some frequently occurring types of ordinary differential equations and the second with the techniques of partial differentiation and solid analytical geometry. The work proper begins with chapter three which considers some of the ways in which partial differential equations arise. They are classified under three headings—formal, geometric and physical processes. After this linear and non-linear equations of the first order are discussed, and before proceeding with partial differential equations of order higher than the first an interruption is occasioned to give acquaintance with the Fourier trigonometry series so that knowledge of this may later be applied. A study of linear equations of second and higher orders, and non-linear equations of second order completes the text. As the book is intended to deal only with the elementary phases of the subject, treatment of these latter topics is therefore restricted.

There are many illustrative worked-out examples throughout the text which should greatly improve the situation of the student. Also, to the same end are many problems for exercise, the answers to which can be found in the back of the

book. A subject index makes for better use of the book for reference. Teachers of mathematics will find this book of value as a text, a clarification of a space in the subject that needs further and more definite understanding.

R. H. OPPERMANN.

THIS PHYSICAL WORLD, by C. C. Clark and C. A. Johnson, 528 pages, illustrations, 16 × 24 cms. McGraw-Hill Book Company, Inc., New York, 1941. Price, $3.25.

This work is the second volume of the two volume series "A College Course in General Science": the companion volume being "This Living World."

It is written in a simple and readable style intended for the student who has not specialized in any branch of science. There are fifteen chapters well illustrated with halftone and colored plates. Each chapter is ended with a well arranged list of references for further studying the subjects. The authors have presented each chapter in a delightful form, discussing the nature of scientific laws and the application of science to astronomy, physics and chemistry.

W. A. R. PERTUCH.

CURRENT TOPICS.

First Concrete Paving 50 Years Old.—TRACY BARTHOLOMEW in *Engineering News Record*, Vol. 127, No. 9, states that fifty years of continuous service by the first concrete street pavement in America was the important event in civil engineering history celebrated in Bellefontaine, Ohio, recently with the unveiling of an appropriate concrete monument. The four concrete pavements surrounding the court house in the center of Bellefontaine have all been in continuous service for from 47 to 50 years. Constructed without modern machinery for manufacturing the cement, mixing the concrete, or laying the pavement—indeed even without the benefit of previous experience by others, or standards to guide the work—these pavements have already outlived most of the men who built them as well as hundreds of other concrete pavements which were built much later. They have both historic and technical interest and bear incontrovertible evidence both of the fundamental knowledge and integrity of their builders, and of the permanence of well-made concrete. Probably few other street pavements have ever given such long and satisfactory service at such negligible expense for maintenance. These streets were planned and built in truly horse-and-buggy days: ten years before the day of the automobile, when heavy drays and pleasure vehicles alike ran on tires of steel. Bellefontaine, in those days, was a prosperous county seat surrounded by rich farm lands widely known for their heavy Percheron horses. It boasted a thriving factory making hames (even the word is almost unknown now) and another making carriages. Just outside the town, an energetic newcomer by the name of Bartholomew was developing a market for a gray powder he called Buckeye portland cement. The pavements are undoubtedly the direct result of the thorough studies, sound conclusions and effective efforts of this man.

R. H. O.

The Trigger of Our Industrial Gun. (*Stone and Webster Bulletin*, Vol. 9, No. 3.)—The earth provides all the things that are needed for human existence, but whether to discipline us or through mere capriciousness, Mother Nature makes it very inconvenient for us at times to obtain the substances we need in the form we want them. Converting materials from one molecular form to another, changing the characteristics of raw materials, and imparting

new qualities to various substances are main activities of the chemical industry. The chemical industry has long been established in the United States. John Winthrop had a factory in Boston as early as 1635 in which he produced saltpeter to make the Colonies independent from other sources of one of the essentials in the production of gunpowder. Some engineers may not consider this a true example of the chemical industry, but only an extractive process in which our country for many years has occupied a preëminent position. By chemical industry is usually meant the business of manufacturing chemicals, as distinguished from the processing industries, such as the refining of petroleum and the manufacture of paper, rayon, glass and plastics. These industries make wide use of chemicals, but do not manufacture them. At the beginning of this century the output of the chemical industry was valued at about one hundred million dollars. Today the annual production exceeds in value a billion and a quarter dollars. Of course, in these years, the other industrial activities of the countries have grown also, but it is interesting to note that at the beginning of the century the chemical industry contributed only nine-tenths of one per cent. to all goods manufactured, while today its contribution exceeds two per cent. This is but a small fraction of the total of the country's manufacturing activities, but it is indispensable to other industries. A trigger is a rather insignificant piece of metal, but its absence from a gun in the hands of a soldier would be very embarrassing. The chemical industry is the trigger of our industrial gun.

R. H. O.

Smokeless Powder Material Increased by Cotton Research.—Defense supplies of smokeless powder for big guns, anti-aircraft shells and army rifles may be made from staple cotton instead of cotton linters, as a result of U. S. Department of Agriculture research. The linters, short fuzzy fibers removed from cotton seed after the ginning process, are particularly adapted to nitrating for the powder, but the present demand for this purpose and for certain other uses is far beyond the supply. The Bureau of Agricultural Chemistry and Eingineering, at its Southern Regional Research Laboratory, found that good-quality, clean cotton fiber of lengths ordinarily used in fabrics, when cut to suitable length, apparently was as well adapted to the powder-making process as the linters. The research men have designed machines for processing the cotton cheaply and quickly and some such machines have already been built. One of these machines operates like a giant lawn mower and cuts batts of cotton four inches thick at the rate of 2400 cuts a minute or two tons of cotton an hour. A second machine, of

different type, cuts the cotton down still farther to the best length for nitration. The short pieces of fiber thus prepared can be readily pumped through the various processes in liquid suspension without "roping" or "spinning," the difficulties that literally and figuratively "tied up" production when attempts were made to use the longer fibers for this purpose. The cotton prepared by these machines for nitrating can hardly be distinguished from the linters that have been found so satisfactory. Quantities of the cut cotton are now being prepared for commercial test runs in the manufacture of smokeless powder. The new source of material indicates the possibility that the powder demand for linters can be met either with linters or with cut cotton and there need not be interference with supplies of suitable material for other purposes such as for cellulose acetate plastics, and nitro-cellulose lacquers.

R. H. O.

Alignment of Cellulose Strands is Key to Cotton Fiber Strength.—A new test method using an X-ray "eye" to look inside cotton fibers has been developed by scientists of the U. S. Department of Agriculture. DR. EARL E. BERKLEY, cotton technologist, explains that cotton fibers have a tube-like construction roughly similar to that of an ordinary paper drinking straw flattened out. The fiber walls are made up of many spiral layers of crystalline cellulose strands. The direction of spiral of any one layer is not continuous but is reversed several times along the length of the fiber. Because of the spiral twist of the cellulose strands, the crystals are aligned at variable angles to the longitudinal axis of the fibers. Repeated tests have demonstrated that the more nearly parallel the strands are to the long axis of the fiber, the stronger is the cotton. Dr. Berkley's X-ray unit makes a photographic diffraction pattern of a bundle of cotton fibers. He then measures the average alignment of the cellulose crystallites in relation to the fiber axis, as shown by the pattern. The measuring is done with a micro-photometer, which is a photoelectric cell device. Most of the angles measured vary from about 25 degrees to 45 degrees, depending on the variety of cotton and the conditions under which it was grown. Fiber strength varies inversely with these angles. Undeteriorated cotton with a 45 degree angle breaks under a pull of about 60,000 pounds per square inch and that with a 25 degree angle breaks at about 95,000 pounds per square inch. X-ray tests are particularly valuable to cotton breeders because they indicate the potential strength of the cotton fiber. Bundle breaking tests cannot do this satisfactorily in all cases because deterioration from bacterial decay or

other causes may greatly weaken the fibers of the sample used. Such deterioration does not affect the alignment angle of the cellulose particles.

R. H. O.

The World's Oddest Hydro Development.—(*Power Plant Engineering*, Vol. 45, No. 8.) Santos, Brazil, the Coffee Port of the World, has another claim to fame—the nearby Serra hydro-electric development, the largest in South America, a potential rival of Grand Coulee. Basically the development consists of a series of dams near the divide, which reverse the natural flow of rivers and send the water over the divide to be utilized in a high head plant. The reservoirs are tied together by natural river beds, canals and two low head pumping stations in such a way as to conserve the water used in an older low head regulating plant, and, to store flood waters from streams outside the present reservoir area. These pumping stations are reversible, to act as emergency generating capacity. Reversed operation will also be used in the dry season to assist in carrying the day load, then at night repumping the water thus discharged so as to augment the normal stream flow. The potential capacity of the entire development is approximately 2,000,000 hp., almost equal to twice the present total installed generating capacity in the whole of Brazil. It is a unique development made possible only by unusual geographic and climatic conditions. The city is supplied with power from hydro stations, the oldest of which is the Parnahyba dam and power house on the Tiete River just below Sao Paulo, built in 1901. The Guarapiranga Reservoir was built in 1908 south of the city on the Pinheiros River to provide regulation for the first plant as well as for two later ones farther downstream on the Tiete. After a severe drought in 1923, A. W. K. Billings, an American engineer was sent to investigate. He conceived the idea of forming interconnecting reservoirs to back the water up over the divide and utilize the drop of almost 2400 feet to the crastal plane below.

R. H. O.

Mineral Oil Deterioration.—J. C. BALSBAUGH, A. G. ASSAF, AND W. W. PENDLETON. (*Industrial and Engineering Chemistry*, Vol. 33, No. 10.) For several years the deterioration of mineral oils in the presence or absence of either copper or paper (or both) has been studied in the laboratories of the Massachusetts Institute of Technology, Cambridge, Mass. This work has been continued with an aim toward interpreting the mechanism involved in the

deterioration of these oils and the roles played by the various types of hydrocarbons in the oil with respect to their electrical and chemical stabilities. A study has been made of (*a*) the effect of temperature on the oxidation characteristics of an oil sample with a viscosity of 100 Saybolt Universal seconds at 100 deg. F., (*b*) the continuous oxidation of a series of related samples having viscosities comparable to those of cable saturants, (*c*) the effect of a limited amount of oxygen on a water-white aromatic-free oil, and (*d*) the oxidation and electrical characteristics of cetane and cis-Decalin. The results show that up to 95 deg. C. the mechanism of deterioration was approximately the same. The series of related oils displayed varying electrical and chemical stabilities with varying aromaticity. A limited amount of oxygen produced higher electrical losses than those obtained with an unlimited supply; and the hydrocarbons exhibited widely divergent chemical and electrical properties.

R. H. O.

Quiet, Please.—GEORGE METZ in *Power*, Vol. 85, No. 10, states that a minor item in a quarter-million-kw. construction program, sound-protected telephone booths are necessary to efficient communication amid the high-pitched hum of 3600-r.p.m. turbines. With a few modifications, a standard steel booth with 1½ in. rockwool lining covered by a protective surface of sheet metal has been reported comfortable and convenient by the operating staff. Jointly developed by the manufacturer, the New York Telephone Co. and the Consolidated Edison Co., the booth was designed for noise levels up to 100 decibels, for the accommodation of several instruments and for both audible and visible signals. In addition, general durability and reasonable moisture resistance were required. The folding-door booth was selected after tests by the operators who were eventually to use the phones. Sound-level readings taken within the booth with the door open and closed indicated only a slight reduction in decibel-level when the door was closed. To the human ear, however, the door closing seemed to have a pronounced effect upon the comfort with which conversation could be carried on. Another consideration in selecting the standard door-type booth was its compactness, space being somewhat limited.

R. H. O.

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